# On the theory of spherically symmetric thin shells in Conformal Gravity

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# I.1 General scheme

## Preliminaries

- Thin shell = 3-dim hypersurface,
  where the energy-momentum tensor is concentrated

$$T_{\mu\nu}\Big|_{\Sigma} = S_{\mu\nu}\delta(\Sigma) \quad \Rightarrow \quad \text{Dirac's } \delta\text{-function}$$

  Here we will consider only timelike shells

- Spherical symmetry $\to$ simplest generalization of a point mass
  Main advantage — backreaction $\to$ self-consistency
  Metric $(\mu, \nu = 0, 1, 2, 3)$, $(i, k = 0, 1)$ :

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ik}dx^i dx^k - r^2(x)(d\theta^2 + \sin^2\theta d\phi^2)$$

- Conformal transformation

$$ds^2 = \Omega^2 d\hat{s}^2 = r^2\left(\tilde{\gamma}_{ik}dx^i dx^k - (d\theta^2 + \sin^2\theta d\phi^2)\right)$$

- $(2+2)$ — decomposition

## I.2 Some formulae

- Riemann curvature tensor: $R^{\mu}_{\nu\lambda\sigma}$
- Ricci tensor: $R^{\mu}_{\nu} = R^{\lambda}_{\mu\lambda\nu}$
- Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
- Weyl tensor (completely traceless):

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \frac{1}{2}(R_{\mu\sigma}g_{\nu\lambda} + R_{\nu\lambda}g_{\mu\sigma} - R_{\mu\lambda}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\lambda}) + \frac{1}{6}R(g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

- Bach tensor: $B_{\mu\nu} = C_{\mu\lambda\nu\sigma}{}^{;\sigma;\lambda} + \frac{1}{2} R^{\lambda\sigma} C_{\mu\lambda\nu\sigma}$

$$B^{\lambda}_{\lambda} = 0, \quad B_{\mu\nu} = B_{\nu\mu}, \quad B^{\lambda}_{\mu;\lambda} = 0$$

# I.3 Conformal transformation and $(2+2)$ – decomposition

- Einstein tensor

$$G_{\mu\nu} = \hat{G}_{\mu\nu} - \frac{2r_{\mu;\nu}}{r} + \frac{2r^{\lambda}{}_{;\lambda}r}{r}\hat{g}_{\mu\nu} + \frac{4r_\mu r_\nu}{r^2} - \frac{r^\lambda r_\lambda}{r^2}\hat{g}_{\mu\nu}$$

$$g_{\mu\nu} = r^2\hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{r^2}\hat{g}^{\mu\nu}, \quad r_\mu = r_{,\mu}, \quad r^\lambda = \hat{g}^{\lambda\sigma}r_{,\sigma}$$

";" — covariant derivative with respect to $\hat{g}_{\mu\nu}$

$$G_{ik} = -\frac{2r_{i|k}}{r} + \frac{4r_i r_k}{r^2} + \left(1 + \frac{2r^p{}_{|p}}{r} - \frac{r^p r_p}{r^2}\right)\tilde{\gamma}_{ik},$$

$$G = G^\lambda_\lambda = -R = -\frac{1}{r^2}\left(-\hat{R} + \frac{6r^l{}_{|l}}{r}\right), \quad \hat{R} = \tilde{R} - 2$$

$\tilde{R}$ — scalar curvature of 2-dim space-time with the metric $\tilde{\gamma}_{ik}$
"|" — covariant derivative with respect to $\tilde{\gamma}_{ik}$

# I.4 Conformal transformation and $(2+2)$ – decomposition

- Bach tensor

$$B_{\mu\nu} = \frac{1}{r^2}\hat{B}_{ik}$$

$$\hat{B}_{ik} = -\frac{1}{6}\left(\tilde{R}^{|p}_{|p}\tilde{\gamma}_{ik} - \tilde{R}_{|ik} + \frac{\tilde{R}^2 - 4}{4}\tilde{\gamma}_{ik}\right)$$

$$\hat{B}^{\mu}_{\mu} = 0 \qquad \Rightarrow \qquad \hat{B}^2_2 = \hat{B}^3_3 = -\frac{1}{2}\hat{B}^l_l$$

# I.5 Gauss normal coordinates

$$ds^2 = \Omega^2 d\hat{s}^2, \quad d\hat{s}^2 = -dn^2 + g_{ij} dx^i dx^j$$

- Hypersurface $\Sigma$: $\quad n = 0$
- Extrinsic curvature tensor:

$$\hat{K}_{ij} = -\frac{1}{2}\frac{\partial \hat{g}_{ij}}{\partial n}$$

- Spherical symmetry $\Rightarrow$

$$ds^2 = r^2 \left( \tilde{\gamma}_{ik} dx^i dx^k - (d\theta^2 + \sin^2\theta d\phi^2) \right)$$

$$d\hat{s}_2^2 = \tilde{\gamma}_{ik} dx^i dx^k = \tilde{\gamma}_{00}(\tau, n) d\tau^2 - dn^2$$

$\tilde{\gamma}_{00}(\tau, 0) = 1 \quad \Rightarrow \quad \tau$ — "conformal proper time" on the shell

$$\hat{K}_{00} = -\frac{1}{2}\frac{\partial \tilde{\gamma}_{00}}{\partial n} = \tilde{K}_{00}, \quad \tilde{K} = \tilde{K}_0^0 = -\frac{1}{2}\frac{\partial \log \tilde{\gamma}_{00}}{\partial n}$$

- $2 - dim$ scalar curvature: $\quad \tilde{R} = -2\tilde{K}_n + 2\tilde{K}^2$

# I.6 Energy-momentum tensor

$$\delta S_{\mathrm{matter}} \stackrel{def}{=} \frac{1}{2}\int T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\,dx$$

$$\delta S_{\mathrm{matter}} \stackrel{def}{=} \frac{1}{2}\int \hat{T}_{\mu\nu}\sqrt{-\hat{g}}\,\delta \hat{g}^{\mu\nu}\,dx$$

$$\hat{T}_{\mu\nu} = r^2 T_{\mu\nu}, \quad \hat{T}^{\nu}_{\mu} = r^4 T^{\nu}_{\mu}, \quad \hat{T}^{\mu\nu} = r^6 T^{\mu\nu}$$

$$\boxed{\hat{T}_{\mu\nu} \stackrel{def}{=} \hat{S}_{\mu\nu}\delta(n) + [\hat{T}_{\mu\nu}]\Theta(n) + \hat{T}^{(-)}_{\mu\nu}}$$

$\hat{S}_{\mu\nu}$ — surface energy-momentum tensor
$\delta(n)$ — Dirac's $\delta$–function
$\Theta(n)$ — Heaviside step function

$$\Theta(n) = \begin{cases} 1, & \text{if } n > 0 \ (+) \\ 0, & \text{if } n < 0 \ (-) \end{cases}$$

$\Theta^2 = \Theta, \ \ \Theta'(n) = \delta(n)$
$[\ldots] =$ "jump" $\ \Rightarrow \ [\hat{T}_{\mu\nu}] = [\hat{T}^{(+)}_{\mu\nu} - \hat{T}^{(-)}_{\mu\nu}]$

# The End of Chapter I

## (to be continued)

# II.1 Einstein Gravity = General Relativity

- Action integral

$$S = \frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\,dx + S_{\mathrm{matter}}$$

- Equations of motion

$$\frac{1}{8\pi G}(G_{\mu\nu} - \Lambda g_{\mu\nu}) = T_{\mu\nu}$$

- Spherical symmetry

$$(00) \quad \Rightarrow \quad -\frac{2r_{,nn}}{r}\tilde{\gamma}_{00} + \frac{r_{,n}^2}{r}\tilde{\gamma}_{00} + \frac{3\dot{r}^2}{r^2} + 1 = \frac{8\pi G}{r^2}\hat{T}_{00}$$

$$(0n) \quad \Rightarrow \quad -\frac{2}{r}(\dot{r}_{,n} + \tilde{K}\dot{r}) + \frac{4\dot{r}r_{,n}}{r^2} = \frac{8\pi G}{r^2}\hat{T}_{0n}$$

$$(nn) \quad \Rightarrow \quad \frac{3}{r^2}r_{,n}^2 - 1 + \frac{2}{\tilde{\gamma}_{00}}\frac{\ddot{r}}{r} + \frac{\dot{\tilde{\gamma}}_{00}}{\tilde{\gamma}_{00}^2}\frac{\dot{r}}{r} + \frac{\dot{r}^2}{\tilde{\gamma}_{00}r^2} - \frac{2\tilde{K}r_{,n}}{r} = \frac{8\pi G}{r^2}\hat{T}_{nn}$$

$$(Tr) \Rightarrow 1 + \tilde{K}_{,n} - \tilde{K}^2 + \frac{3}{r}\left(\frac{\ddot{r}}{\tilde{\gamma}_{00}} - \frac{1}{2}\frac{\dot{\tilde{\gamma}}_{00}}{\tilde{\gamma}_{00}^2}\dot{r} + \tilde{K}r_{,n} - r_{,nn}\right) = \frac{4\pi G}{r^2}Tr\,\hat{T} + 2\Lambda r^2$$

# II.2 Einstein Gravity $=$ General Relativity

Metric tensor is continuous $\Rightarrow$

$$\begin{cases} [r] = 0, \; r_{,n} = [r_{,n}]\Theta(n) + r_{,n}^{(-)} \\ r_{,nn} = [r_{,n}]\delta(n) + [r_{,nn}]\Theta(n) + r_{,nn}^{(-)} \end{cases}$$

$$\begin{cases} [\tilde{\gamma}_{00}] = 0, \; \tilde{K} = [\tilde{K}]\Theta(n) + \tilde{K}^{(-)} \\ \tilde{K}_{,n} = [\tilde{K}]\delta(n) + [\tilde{K}_{,n}]\Theta(n) + \tilde{K}_{,n}^{(-)} \end{cases}$$

No $\delta'(n)$ $\Rightarrow$ double layer is not allowed in GR!

Matching conditions on $\Sigma$ $=$ Einstein equations on $\Sigma$ $=$ Israel equations:

$$\begin{aligned} (\ldots)\delta(n) &= (\ldots)\delta(n) \\ (\ldots)\Theta(n) &= (\ldots)\Theta(n) \end{aligned}$$

$(\Theta^2 = \Theta, \; \tilde{\gamma}_{00}(\tau, 0) = 1)$

# II.3 Einstein Gravity $=$ General Relativity

$$(00) \;\Rightarrow\; \begin{cases} -\frac{2}{r}[r_{,n}] = \frac{8\pi G}{r^2}\hat{S}_{00} \\ -\frac{1}{r^2}[r_{,n}^2] - \frac{2}{r}[r_{,nn}] = \frac{8\pi G}{r^2}\hat{T}_{00} \end{cases}$$

$$(0n) \;\Rightarrow\; \begin{cases} \hat{S}_{0n} = 0 \\ -\frac{2}{r}[r_{,n}] - \frac{2\dot{r}}{r}[\tilde{K}] + \frac{4\dot{r}}{r^2}[r_{,n}] = \frac{8\pi G}{r^2}\hat{T}_{0n} \end{cases}$$

$$(nn) \;\Rightarrow\; \begin{cases} \hat{S}_{nn} = 0 \\ \frac{3}{r^2}[r_{,n}] - \frac{2}{r}[\tilde{K} r_{,n}] = \frac{8\pi G}{r^2}\hat{T}_{nn} \end{cases}$$

$$(Tr) \;\Rightarrow\; \begin{cases} [\tilde{K}] - \frac{3}{r}[r_{,n}] = \frac{4\pi G}{r^2}(Tr\,\hat{S}) \\ [\tilde{K}_{,n}] - [\tilde{K}^2] + \frac{3}{r}([\tilde{K} r_{,n}] - [r_{,nn}]) = \frac{4\pi G}{r^2}(Tr\,\hat{T}) \end{cases}$$

# II.4 Einstein Gravity $=$ General Relativity

- Shell radius: $\rho = r(\tau, 0) = r_{|\Sigma}$

$$\begin{cases} \dot{\hat{S}}_0^0 - \frac{\dot{\rho}}{\rho}(\hat{S}_0^0 + 2\hat{S}_2^2) + [\hat{T}_0^n] = 0 \\ -[r_{,n}] = \frac{4\pi G}{\rho^2}\hat{S}_0^0 \\ -[\tilde{K}] = \frac{8\pi G}{\rho^2}(\hat{S}_0^0 - \hat{S}_2^2) \end{cases}$$

First equation is the shell's conservation equation

Note: if $[\tilde{K}]_{|\Sigma} \neq 0$ (and so $\hat{S}_0^0 \neq \hat{S}_2^2$) $\Rightarrow$

$\tilde{R}$ is singular on the shell

- Introduce the invariant $\Delta = \tilde{\gamma}^{ik} r_{,i} r_{,k} = \tilde{\gamma}^{00} \dot{r}^2 - r_{,n}^2 \ \Rightarrow$

$$r_{,n}{}_{|\Sigma} = \sigma\sqrt{\dot{\rho}^2 - \Delta}, \quad \sigma = \pm 1$$

# II.5 Einstein Gravity = General Relativity

- Traceless shell in vacuum $\Rightarrow$ $\hat{S}^2_2 = -\frac{1}{2}\hat{S}^0_0$, $\hat{T}^{(+)}_{\mu\nu} = T^{(-)}_{\mu\nu} = 0$

$$\begin{cases} \dot{\hat{S}}^0_0 = 0 \\ -[r_{,n}] = \frac{4\pi G}{\rho^2}\hat{S}^0_0 \\ -[\tilde{K}] = \frac{12\pi G}{\rho^2}\hat{S}^0_0 \end{cases}$$

- Birkhoff theorem: $\Delta - \rho^2\left(1 - \frac{2Gm}{\rho} - \frac{\Lambda}{3}\rho^2\right)$

$\hat{S}^0_0 = const = \hat{S}_0$, $m_{\rm in} = 0$, $m_{\rm out} = 0$ — from initial conditions

$$\begin{cases} \sigma_{\rm in}\sqrt{\dot{\rho}^2 - \Delta_{\rm in}} - \sigma_{\rm out}\sqrt{\dot{\rho}^2 - \Delta_{\rm out}} = \frac{4\pi G}{\rho^2}\hat{S}_0 \\ -[\tilde{K}] = \frac{12\pi G}{\rho^2}\hat{S}^0_0 \neq 0 \end{cases}$$

- $\sigma_{\rm in}$ and $\sigma_{\rm out}$ — global geometry
- Solution: $\tilde{R}_{\rm in} = 2 \;\Rightarrow\; 2 = -2\tilde{K}_{,n} + 2\tilde{K}^2 \;\Rightarrow$

$\tilde{K} = -\tanh(n + \varphi(\tau))$, $\quad \tilde{\gamma}_{00} = \frac{\cosh^2(n+\varphi(\tau))}{\cosh^2(\varphi(\tau))}$

Or, yet another solution:

$\tilde{K} = -\coth(n + \varphi(\tau))$, $\quad \tilde{\gamma}_{00} = \frac{\sinh^2(n+\varphi(\tau))}{\sinh^2(\varphi(\tau))}$

In what follows we will need only the first one

# The End of the Chapter II

(to be continued)

# III.1 Weyl Gravity $=$ Conformal Gravity

Action integral:

$$S = -\alpha_0 \int C^2 \sqrt{-g}\, dx + S_{\mathrm{matter}}$$

$C^2 = C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$, $C_{\mu\nu\lambda\sigma}$ — Weyl tensor

$$\begin{aligned} C_{\mu\nu\lambda\sigma} &= R_{\mu\nu\lambda\sigma} + \frac{1}{2}(R_{\mu\sigma}g_{\nu\lambda} + R_{\nu\lambda}g_{\mu\sigma} - R_{\mu\lambda}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\lambda}) \\ &\quad + \frac{1}{6}R(g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\sigma}g_{\nu\lambda}) \end{aligned}$$

$C^{\mu}_{\nu\lambda\sigma} = \hat{C}^{\mu}_{\nu\lambda\sigma}$ — conformally invariant

$C^2\sqrt{-g} = \hat{C}^2\sqrt{-\hat{g}}$ — conformally invariant in $dim = 4$

Equations of motion (= Bach equations): $\boxed{8\alpha_0 B_{\mu\nu} = T_{\mu\nu}}$

Bach tensor: $B_{\mu\nu} = C_{\mu\lambda\nu\sigma}{}^{;\sigma;\lambda} + \frac{1}{2} R^{\lambda\sigma} C_{\mu\lambda\nu\sigma}$

$B^{\lambda}_{\lambda} = 0$, $B_{\mu;\lambda}^{\;\lambda} = 0 \;\Rightarrow\; T^{\lambda}_{\lambda} = 0$, $T_{\mu;\lambda}^{\;\lambda} = 0$

$B_{\mu\nu} = \frac{1}{\Omega^2}\hat{B}_{\mu\nu}$, $T_{\mu\nu} = \frac{1}{\Omega^2}\hat{T}_{\mu\nu}$

# III.2 Weyl Gravity

## Conformal transformation and $(2+2)$ – decomposition

- Spherical symmetry: $B_{\mu\nu} = -\frac{1}{r^2}\hat{B}_{\mu\nu}$, $\hat{B}^{\mu}_{\mu} = 0$, $B^2_2 = B^3_3 = -\frac{1}{\alpha_0}B^l_l$

$$\hat{B}_{ik} = -\frac{1}{6}\left(\tilde{R}^{|p}_{|p}\tilde{\gamma}_{ik} - \tilde{R}_{|ik} + \frac{\tilde{R}^2 - 4}{4}\tilde{\gamma}_{ik}\right) = \frac{1}{8\alpha_0}\hat{T}_{ik}$$

"|" — covariant derivative with respect to $\tilde{\gamma}_{ik}$

$\hat{B}^{\mu}_{\mu} = 0 \;\Rightarrow\; B^2_2 = B^3_3 = -\frac{1}{\alpha}B^l_l$

- $\tilde{R}$ — scalar curvature of 2-dim space-time with the metric $\tilde{\gamma}_{ik}$, contains second derivatives of the metric tensor
- We need no more equations
- No trace of the radius $r$

Gauss normal coordinate system:

$$\begin{cases} \hat{B}_{00} = \frac{1}{6}\left(\frac{\tilde{R}^2-4}{4} - \tilde{R}_{,nn}\right)\tilde{\gamma}_{00} = \frac{1}{8\alpha_0}\hat{T}_{00} \\ \hat{B}_{0n} = -\frac{1}{6}\left(\dot{\tilde{R}}_{,n} + \tilde{K}\dot{\tilde{R}}_{,n}\right) = \frac{1}{8\alpha_0}\hat{T}_{0n} \\ \hat{B}_{nn} = \frac{1}{6\tilde{\gamma}_{00}}\left(\ddot{\tilde{R}} - \frac{\dot{\tilde{\gamma}}_{00}}{\tilde{\gamma}_{00}}\dot{\tilde{R}} + \tilde{\gamma}_{00}\tilde{K}\tilde{R}_{,n}\right) - \frac{\tilde{R}^2-4}{24} = \frac{1}{8\alpha_0}\hat{T}_{00} \end{cases}$$

# III.3 Weyl Gravity

- The double layer (= something proportional to $\delta^{'}(n)$) can appear only in $\tilde{R}_{,nn}$ — term in (00)– equation.
Since $\delta^{'}(n)$) is not concentrated on $\Sigma$, its consideration requires more generial technics.
Here we suppose that the doulble layer is absent.
- Energy-momentum tensor

$$\hat{T}_{\mu\nu} = \hat{S}_{\mu\nu}\delta(n) + [\hat{T}_{\mu\nu}]\Theta(n) + \hat{T}^{(-)}_{\mu\nu}$$

$$[\tilde{R}] = 0,\ \tilde{R}_{,n} = [\tilde{R}_{,n}]\Theta(n) + \tilde{R}^{(-)}_{,n},\ [\tilde{R}] = 0$$
$$\tilde{R}_{,nn} = [\tilde{R}_{,n}]\delta(n) + [\tilde{R}_{,nn}]\Theta(n) + \tilde{R}^{(-)}_{,nn}$$
$$\tilde{R} = -2\tilde{K}_{,n} + \tilde{K}^2 \ \Rightarrow\ [\tilde{K}] = 0,\ [\tilde{K}_{,n}] = 0$$

- The shell must be traceless! $\Rightarrow$ $\hat{S}^{\lambda}_{\lambda} = 0$

# III.4 Weyl Gravity

$$\begin{cases} -[\tilde{R}_{,n}] = \frac{3}{4\alpha_0}\hat{S}_{00} \\ -[\tilde{R}_{,nn}] = \frac{3}{4\alpha_0}\hat{T}_{00} \end{cases}$$

$$\begin{cases} S_{0n} = 0 \\ -[\dot{\tilde{R}}_{,n}] = \frac{3}{4\alpha_0}\hat{T}_{0n} \end{cases}$$

$$\begin{cases} S_{nn} = 0 \\ -\tilde{K}[\tilde{R}_{,n}] = \frac{3}{4\alpha_0}\hat{T}_{nn} \end{cases}$$

Or,

$$\begin{cases} \dot{\tilde{S}}^0_0 - [\hat{T}^n_0] = 0 \\ -[\tilde{R}_{,n}] = \frac{3}{4\alpha_0}\hat{S}^0_0 \\ -\tilde{K}[\tilde{S}^0_0] + [\hat{T}^n_n] = 0 \end{cases}$$

# III.5 Weyl Gravity

- The (traceless) shell in vacuum: $\hat{S}^0_0 = \hat{S}_0 = const$

$$\begin{cases} -[\tilde{R}_{,n}] = \frac{3}{4\alpha_0}\hat{S}^0_0 = const \neq 0 \\ -\tilde{K}_{|\Sigma} = \tilde{K}(\tau, 0) = 0 \end{cases}$$

- No trace of radius
- Introduce the invariant $\tilde{\Delta} = \tilde{\gamma}^{ik}\tilde{R}_{,i}\tilde{R}_{,k}$

On the shell

$$\tilde{R}_{,n} = \tilde{\sigma}\sqrt{\dot{\tilde{R}}^2 - \tilde{\Delta}}, \quad \tilde{\sigma} = \pm 1$$

$$\tilde{\sigma}_{\rm in}\sqrt{\dot{\tilde{R}}^2 - \tilde{\Delta}_{\rm in}} - \tilde{\sigma}_{\rm out}\sqrt{\dot{\tilde{R}}^2 - \tilde{\Delta}_{\rm out}} = \frac{3}{4\alpha_0}\hat{S}_0$$

- Vacuum solution inside the shell.

No more gravitational source at $n < 0$!

$\tilde{R} = \pm 2 \;\Rightarrow\; \dot{\tilde{R}} = 0, \;\; \tilde{\Delta}_{\rm in} = 0 \;\Rightarrow$

$-\tilde{\sigma}_{\rm out}\sqrt{-\tilde{\Delta}_{\rm out}} = \frac{3}{4\alpha_0}\hat{S}_0 = const \;\Rightarrow$

$$\tilde{\Delta}_{{\rm out}|\Sigma} = const$$

# III.6 Weyl Gravity

- Solution at $\tilde{R} = +2$

$\tilde{R} = -2\tilde{K}_{,n} + 2\tilde{K}^2 = 2 \;\Rightarrow\; \tilde{K} = -\tanh(n + \varphi(\tau))$

$\tilde{K}_{|\Sigma} = \tilde{K}(\tau, 0) = 0 \;\Rightarrow\; \varphi(\tau) = 0 \;\Rightarrow\; \tilde{K} = -\tanh n$

$\tilde{\gamma}_{00-} = \cosh^2 n, \quad (\tilde{\gamma}_{00}(\tau, 0) = 1)$

$d\tilde{s}_2^2 = \cosh^2 n\, d\tau^2 - dn^2$

- Solution at $\tilde{R} = -2$

$\tilde{R} = -2\tilde{K}_{,n} + 2\tilde{K}^2 = -2 \;\Rightarrow\; \tilde{K} = \tanh n$

$\tilde{K}_{|\Sigma} = \tilde{K}(\tau, 0) = 0 \;\Rightarrow\; \varphi(\tau) = 0 \;\Rightarrow\; \tilde{K} = \tanh n$

$\tilde{\gamma}_{00} = \cos^2 n \;\; (\tilde{\gamma}_{00-}(\tau, 0) = 1)$

$d\tilde{s}_2^2 = \cos^2 n\, d\tau^2 - dn^2$

# III.7 Weyl Gravity

- Vacuum solution outside the shell $(\tilde{R} \neq const)$

$$ds^2 = A d\eta^2 - \frac{1}{A} d\tilde{R}^2$$

$$A = \frac{1}{6}(\tilde{R}^3 - 12\tilde{R} + C_0) = -\tilde{\Delta}_{\mathrm{out}}$$

$$\tilde{\Delta}_{\mathrm{out}\big|\Sigma} = \frac{9}{16\alpha_0^2}\hat{S}_0^2$$

$$\begin{cases} \frac{27}{8\alpha_0^2}\hat{S}_0^2 + 16, & \text{if } \tilde{R}_{\big|\Sigma} = +2 \\ \frac{27}{8\alpha_0^2}\hat{S}_0^2 - 16, & \text{if } \tilde{R}_{\big|\Sigma} = -2 \end{cases}$$

- What is the trajectory of the shell, $\rho(\tau)$?

It is completely arbitrary!

# The End of the Chapter III

## (to be continued)

# IV.1 Weyl + Einstein

- Action integral

Weyl + Einstein

- Equations of motion

Bach + Einstein

- Spherical symmetry

Thin shell and no double layer

Matching conditions:

$$\begin{cases} \dot{\hat{S}}{}^0_0 - \frac{\dot{r}}{r}(\hat{S}^0_0 + \hat{S}^2_2) + \hat{T}^n_0 = 0 \\ -2\alpha_0[\tilde{R}_{,n}] = \hat{S}^0_0 - \hat{S}^2_2 \\ -3r[r_{,n}] = 4\pi G(\hat{S}^0_0 + 2\hat{S}^2_2) \\ -2\alpha_0[\tilde{R}_{,nn}] = \tilde{K}(\hat{S}^0_0 + \hat{S}^2_2) - [\hat{T}^0_0 - \hat{T}^n_n - \hat{T}^2_2] \\ r[r_{,nn}] = -\frac{4\pi G}{3}\left(\tilde{K}(\hat{S}^0_0 + 2\hat{S}^2_2) + Tr[\tilde{T}]\right) \\ -[\Delta] = \frac{8\pi G}{3}(\tilde{K}\hat{S}^0_0 + [\tilde{T}^n_n]) \end{cases}$$

# IV.2 Weyl + Einstein

- Traceless shell $(\hat{S}^0_0 + \hat{S}^2_2) = 0$ in vacuum

$$
\begin{cases}
\dot{\hat{S}}^0_0 = 0 \\
-[\tilde{R}_{,n}] = \frac{3}{4\alpha_0}\hat{S}^0_0 \\
-r[r_{,n}] = 0 \\
4\alpha_0[\tilde{R}_{,nn}] = \tilde{K}\hat{S}^0_0 \\
r[r_{,nn}] = 0 \\
-[\Delta] = \frac{8\pi G}{3}\tilde{K}\hat{S}^0_0
\end{cases}
$$

$$
[r_{,n}] = 0 \;\Rightarrow\; [\Delta] = 0 \;\Rightarrow\; \tilde{K} = 0 \;\Rightarrow\; \boxed{-[\tilde{R}_{,n}] = \frac{3}{4\alpha_0}\hat{S}^0_0 = const}
$$

- The same as in the pure Conformal Gravity
- No trace of the trajectory $\rho(\tau)$
- But!

# IV.3 Weyl + Einstein

- Vacuum equations

$$
\begin{aligned}
- &\quad \frac{4}{3}\alpha_0 \left( \tilde{R}^{|p}_{|p}\tilde{\gamma}_{ik} - \tilde{R}_{|ik}\frac{\tilde{R}^2-4}{4}\tilde{\gamma}_{ik} \right) \\
+ &\quad \frac{1}{8\pi G}\left( 2rr_{i|k} + 4r_i r_k + (2rr^l{}_{|l} - r^l r_l + r^2 - \Lambda r^4)\tilde{\gamma}_{ik} \right) = 0 \\
&\quad 2 - \tilde{R} + \frac{6r^l{}_{|l}}{r} = 4\Lambda r^2
\end{aligned}
$$

- (Anti)De Sitter:

$$\tilde{R} = 2, \quad \Delta = -r^2(1 - \frac{\Lambda}{3}r^2)$$

(Bach=0, Einstein=0)

- Metric in Gauss normal coordinates

$$d\tilde{s}^2 = \cosh^2 n\, d\tau^2 - dn^2$$

- $r(\tau, n) = ? \ \Rightarrow \ \rho(\tau) = r(\tau, 0)$
- 4 equations for 1 function

# IV.4 Weyl + Einstein

- The solution exists!

$$r = \frac{1}{\cosh n\left(C_1 \cos[\tau - \tau_0] + C_2 \sin[\tau - \tau_0]\right) + d_0 |\sinh n|}$$

$$C_1^2 + C_2^2 = \frac{\Lambda}{3} + d_0^2$$

- Trajectory: $n = 0,\ r(\tau, 0) = \rho(\tau)$

$$\boxed{\rho(\tau) = \frac{1}{C_1 \cos[\tau] + C_2 \sin[\tau]} = \frac{1}{C_0 \sin[\tau - \tau_0]}}$$

- But! Don't be afraid of this formula

The "genuine proper time"

$dt = \rho(\tau) d\tau \quad \Longrightarrow \quad t = \int^{\tau_0} \frac{d\tau}{C_0 \sin[\tau - \tau_0]} = \infty$

(!)

# The End of the Chapter IV

(to be continued)

# The End

## Thanks to all