Relativistic anisotropic stars with the polytropic equation of state in general relativity

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Introduction

A compact stellar object with the spherically symmetric distribution of matter can nevertheless be characterized by the local pressure anisotropy:

R. L. Bowers and E. P. T. Liang, Astrophys. J. 188, 657 (1974).

Local pressure anisotropy can be caused by different reasons:

- **presence of strong magnetic fields inside a star**
- **availability of superfluid states with the finite orbital momentum of Cooper pairs or finite superfluid momentum**
- **appearance of spontaneous deformation of Fermi surfaces**
- **existence of a solid core**

The analysis of the generalized equations of hydrostatic equilibrium shows that the pressure anisotropy may have the substantial effect on the maximum equilibrium mass and gravitational surface redshift.

At densities $\rho \gtrsim 10^{18}$ kg/m^3 both the relativistic effects and the effects of general relativity become important.

Introduction

With account of the pressure anisotropy, the equation of state (EoS) of the system will be also necessarily anisotropic. The EoS is the essential ingredient in solving the equations of the hydrostatic equilibrium. In the given work, we choose the polytropic EoS, which is widely used in many astrophysical applications.

In this research, we will study spherically symmetric relativistic anisotropic stars with the polytropic EoS. The generalized Lane-Emden equations will be obtained for the arbitrary anisotropy parameter $\Delta = p_t - p_r$ (p_t and p_r being the transverse and radial pressure, respectively) and then applied to the special ansatz for the anisotropy parameter in the form of the differential relation between the anisotropy parameter Δ and the metric function v.

The analytic solutions of these equations can be found for incompressible anisotropic fluid stars which then used to get their mass-radius relation, gravitational and binding energy. Also, we will apply the Chandrasekar variational procedure to study the dynamical stability of incompressible anisotropic fluid stars with respect to radial oscillations.

Basic Equations

For spherically symmetric stars

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

The anisotropic energy-momentum tensor for the static configuration

$$T_i^k = \mathrm{diag}(\varepsilon, -p_r, -p_t, -p_t)$$

Einstein equations $$R_i^k - \frac{1}{2}R\delta_i^k = 8\pi G T_i^k$$

For the static configuration:

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G\varepsilon,$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi G p_r,$$

$$\frac{1}{2}e^{-\lambda}\left(\nu'' - \frac{1}{2}\nu'\lambda' + \frac{1}{2}\nu'^2 + \frac{\nu' - \lambda'}{r}\right) = 8\pi G p_t$$

Basic Equations

The equation of hydrostatic equilibrium in a spherically symmetric anisotropic star

$$p_r{}' = -G\frac{(\varepsilon + p_r)(m(r) + 4\pi p_r r^3)}{r(r-2Gm(r))} + \frac{2\Delta}{r}, \quad \Delta \equiv p_t - p_r$$

$$m(r) = 4\pi\int_0^r \varepsilon r^2 dr.$$

Boundary conditions

$$p_r(0) = p_{r0} \qquad p_r(R) = 0$$

$$m(0) = 0 \qquad M = m(R)$$

Metric functions

$$e^{-\lambda(r)} = 1 - \frac{2G}{r}m(r), \quad r < R, \qquad \nu' = 2G\frac{m(r) + 4\pi p_r r^3}{r(r-2Gm(r))}.$$

At the boundary *r=R*

$$\lambda(R) = -\nu(R) = -\ln(1 - \frac{2GM}{R})$$

Basic Equations

The polytropic EoS

$$p_r = K\rho^{\gamma} \equiv K\rho^{1+\frac{1}{n}}$$

γ – polytropic exponent, n – polytropic index

The energy density for the polytropic EoS

$$\varepsilon = \rho + \frac{p_r}{\gamma - 1}$$

Lane-Emden function θ

$$p_r = p_{r0}\theta^{n+1}, \quad \rho = \rho_0\theta^n,$$

$$\theta(0) = 1, \quad \theta(R) = 0.$$

Equation of hydrostatic equilibrium for an anisotropic spherical star

$$2q_0(n+1)d\theta - \frac{4\Delta dr}{\rho_0 r\theta^n} + (1+(n+1)q_0\theta)d\nu = 0, \qquad q_0 \equiv \frac{p_{r0}}{\rho_0}$$

Basic Equations

The equation of hydrostatic equilibrium can be integrated:

$$\nu = \nu_0 - \ln\left(\frac{1+(n+1)q_0\theta}{1+(n+1)q_0}\right)^2 + \frac{4}{\rho_0}\int_0^r \frac{\Delta dr}{r\theta^n(1+(n+1)q_0\theta)}, \quad \nu_0 \equiv \nu(0)$$

From the boundary condition at r=R:

$$\nu_0 = \ln\frac{1-\frac{2GM}{R}}{(1+(n+1)q_0)^2} - \frac{4}{\rho_0}\int_0^R \frac{\Delta dr}{r\theta^n(1+(n+1)q_0\theta)}.$$

Finally, the metric function *v(r)* reads

$$\nu(r) = \ln\frac{1-\frac{2GM}{R}}{(1+(n+1)q_0\theta)^2} - \frac{4}{\rho_0}\int_r^R \frac{\Delta dr}{r\theta^n(1+(n+1)q_0\theta)}.$$

The auxiliary function

$$u(r) \equiv \frac{m(r)}{M} = \frac{r}{2GM}(1-e^{-\lambda(r)}),\ u(0)=0, u(R)=1,$$

satisfies the differential equation $Mu' = 4\pi\varepsilon r^2.$

Basic equations

Einstein equation for p_r after substituting $e^{-\lambda}=f(u)$ and v' from the equation of hydrostatic equilibrium:

$$\frac{q_0(n+1)\theta' r}{1+(n+1)q_0\theta}\left(1-\frac{2GM}{r}u\right)+\frac{GMq_0\theta}{1+nq_0\theta}u'+$$
$$+\frac{GM}{r}u-\frac{2\Delta}{\rho_0\theta^n(1+(n+1)q_0\theta)}\left(1-\frac{2GM}{r}u\right)=0.$$

The dimensionless radial coordinate ξ and dimensionless function η

$$r=\alpha\xi,\quad \eta=\frac{M}{4\pi\rho_0\alpha^3}u,\qquad \alpha^2=\frac{q_0(n+1)}{4\pi G\rho_0}$$

Generalized Lane-Emden equations for relativistic anisotropic polytropes

$$\frac{\xi-2(n+1)q_0\eta}{1+(n+1)q_0\theta}\left\{\xi\frac{d\theta}{d\xi}-\frac{2\Delta}{\rho_0 q_0(n+1)\theta^n(1+(n+1)q_0\theta)}\right\}+\eta+q_0\xi^3\theta^{n+1}=0,$$

$$\frac{d\eta}{d\xi}=\xi^2\theta^n(1+nq_0\theta).$$

Boundary conditions: $\theta(0)=1,\ \theta(\xi_R)=0,$

$$\eta(0)=0,\ \eta(\xi_R)=\frac{M}{4\pi\rho_0\alpha^3}$$

Basic Equations

The radius R and mass M can be found as functions of the constants q_0 and *K* in the polytropic EoS:

$$R = R^* q_0^{\frac{1-n}{2}} \xi_R, \; M = M^* q_0^{\frac{3-n}{2}} \eta(\xi_R),$$

$$R^* = \sqrt{\frac{n+1}{4\pi G}} K^{\frac{n}{2}}, \; M^* = \frac{1}{\sqrt{4\pi}} \left(\frac{n+1}{G}\right)^{\frac{3}{2}} K^{\frac{n}{2}}.$$

The mass-radius relation

$$\frac{GM}{R} = (n+1) q_0 \frac{\eta(\xi_R)}{\xi_R}.$$

The total energy of a star

$$E = 4\pi \int_0^R \varepsilon r^2 dr = M^* q_0^{\frac{3-n}{2}} \eta(\xi_R).$$

The proper energy

$$E_0 = 4\pi \int_0^R \varepsilon e^{\frac{\lambda}{2}} r^2 dr = 4\pi \rho_0 \alpha^3 \int_0^{\xi_R} (1 + n q_0 \theta) \frac{\xi^2 \theta^n}{\sqrt{1 - \frac{2 q_0 (n+1)}{\xi} \eta(\xi)}} d\xi.$$

Basic Equations

The gravitational potential energy

$$\Omega = E - E_0 = M^* q_0^{\frac{3-n}{2}} \eta(\xi_R)(1 - \frac{1}{\eta(\xi_R)} \int_0^{\xi_R} (1 + n q_0 \theta) \frac{\xi^2 \theta^n}{\sqrt{1 - \frac{2q_0(n+1)}{\xi} \eta(\xi)}} d\xi).$$

The binding energy: the difference between the energy of the particles scattered to infinity and the total energy of the system

$$E_B = E_{0g} - E, E_{0g} = 4\pi \int_0^R \rho e^{\frac{\lambda}{2}} r^2 dr,$$

$$E_B = M^* q_0^{\frac{3-n}{2}} \eta(\xi_R)(\frac{u_g(\xi_R)}{\eta(\xi_R)} - 1), u_g(\xi) = \int_0^{\xi} \frac{\xi^2 \theta^n}{\sqrt{1 - \frac{2q_0(n+1)}{\xi} \eta(\xi)}} d\xi.$$

Ansatz for the anisotropy parameter Δ

In order to solve the generalized Lane–Emden equations for functions *θ, η* with the corresponding boundary conditions, one needs to specify the anisotropy parameter Δ. Some general properties of spherically symmetric relativistic anisotropic stars can be studied with a certain phenomenological ansatz for the anisotropy parameter. In this work, we use some special ansatz for the anisotropy parameter in the form of the differential relation between the anisotropy parameter and the metric function *ν*. Namely, we will suppose that the presence of the anisotropy parameter doesn't change the general form of Lane-Emden equations for relativistic isotropic stars, but only can change the coefficients in these equations. Specifically, we set the differential relation between Δ and *ν* in the form:

$$-\frac{4\Delta dr}{\rho_0 r\theta^n}+(1+(n+1)q_0\theta)d\nu=(1+\beta q_0\theta)d\nu$$

With this ansatz, one can obtain the metric function *ν* in the form

$$\nu(r)=\ln\frac{1-\frac{2GM}{R}}{(1+\beta q_0\theta)^{\frac{2(n+1)}{\beta}}}.$$

Analytical solutions for incompressible anisotropic stars

The modified Lane-Emden equations, corresponding to the above ansatz:

$$\frac{\xi - 2(n+1)q_0\eta}{1+\beta q_0\theta}\xi\frac{d\theta}{d\xi} + \eta + q_0\xi^3\theta^{n+1} = 0,$$

$$\frac{d\eta}{d\xi} = \xi^2\theta^n(1+nq_0\theta)$$

Look as in the isotropic case, but with that difference that the impact of the anisotropy parameter is reflected in the coefficient β (substituting the multiplier (n+1))

The analytical solutions for incompressible anisotropic fluid stars (ρ=const, n=0):

$$\eta(\xi) = \frac{\xi^3}{3}, \qquad \frac{1+3q_0\theta}{1+\beta q_0\theta} = \pm\frac{1+3q_0}{1+\beta q_0}\left(1-\frac{2q_0}{3}\xi^2\right)^{\frac{3-\beta}{4}}.$$

With account of the boundary conditions

$$\theta(\xi) = \frac{1}{q_0}\frac{(1+3q_0)\left(1-\frac{2q_0}{3}\xi^2\right)^{\frac{3-\beta}{4}} - (1+\beta q_0)}{3(1+\beta q_0) - \beta(1+3q_0)\left(1-\frac{2q_0}{3}\xi^2\right)^{\frac{3-\beta}{4}}}.$$

Analytical solutions for incompressible anisotropic stars

The positive root of θ(ξ)

$$\theta(\xi_R) = 0 \Rightarrow \ \xi_R = \sqrt{\frac{3}{2q_0}[1-(\frac{1+\beta q_0}{1+3q_0})^{\frac{4}{3-\beta}}]}$$

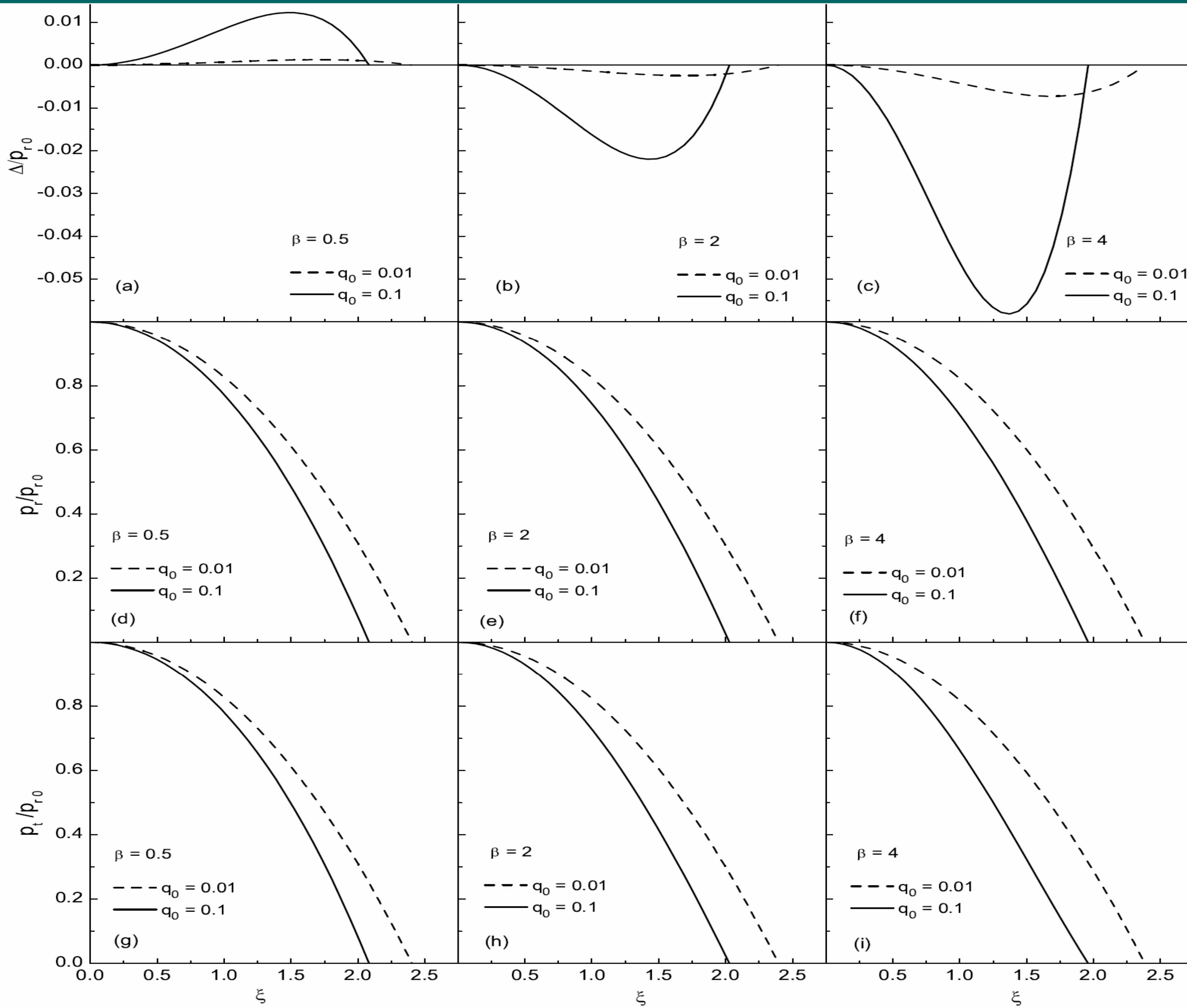
The binding energy of incompressible anisotropic fluid stars

$$E_B = M^*\{-\frac{3}{4}\xi_R\sqrt{q_0(1-\frac{2q_0}{3}\xi_R^2)} + \frac{1}{2}\sqrt{(\frac{3}{2})^3}\arcsin(\sqrt{\frac{2q_0}{3}}\xi_R) - \frac{1}{3}(\sqrt{q_0}\xi_R)^3\}.$$

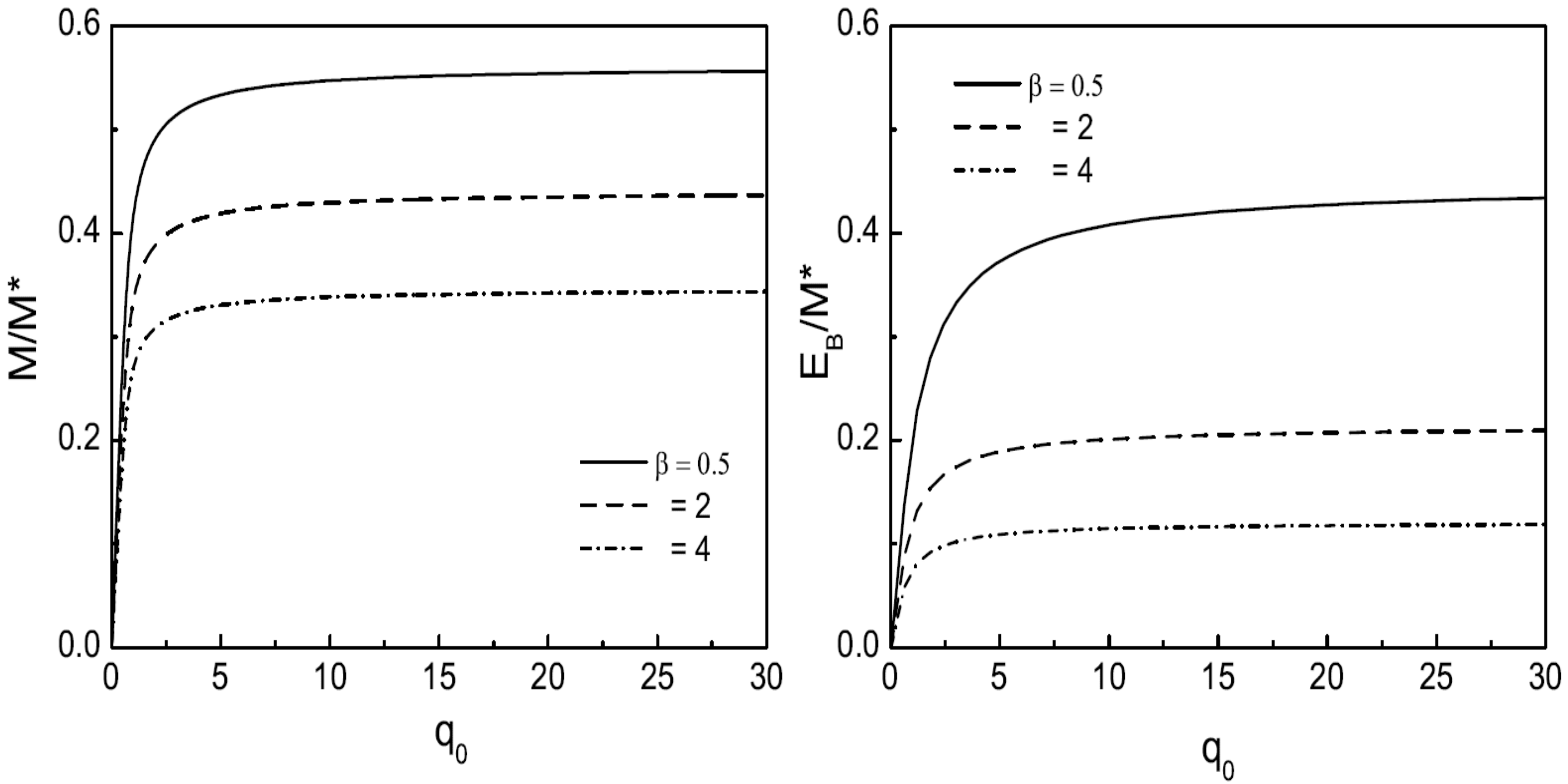
The gravitational potential energy at n=0:

$$\Omega = -E_B$$

$\Delta(\xi)$, $p_r(\xi)$ and $p_t(\xi)$ dependencies



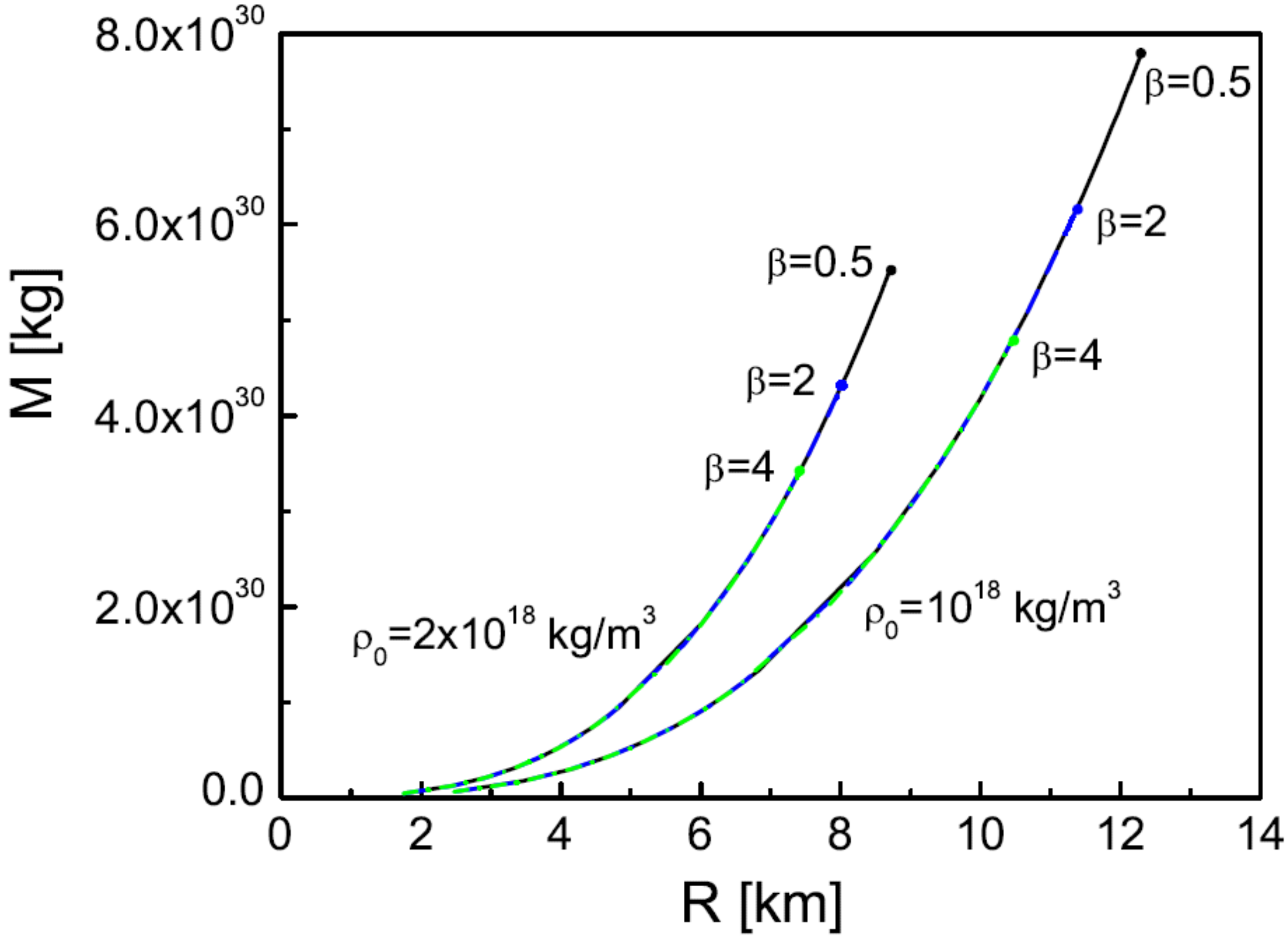
$M(q_0)$, $E_B(q_0)$ and $\Omega(q_0)$dependencies



The gravitational potential energy at n=0:

$$\Omega / M^* = -E_B / M^*$$

Mass-radius relation for incompressible anisotropic stars



The limiting masses for each configuration are shown by full dots. When q_0 varies at the given β, the current point moves along almost the same curve for this specific central density independently of β, the difference being only in the limiting masses for different β.

Dynamical stability of incompressible anisotropic fluid stars

Let us consider the stability of spherically symmetric anisotropic stars with respect to radial oscillations, assuming that they do not violate the spherical symmetry.

Einstein equations:

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=8\pi G T_0^0, \quad \lambda'\equiv\frac{\partial\lambda}{\partial r}$$

$$-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}=8\pi G T_1^1,$$

$$-\frac{1}{2}e^{-\lambda}\left(\nu''-\frac{1}{2}\nu'\lambda'+\frac{1}{2}\nu'^2+\frac{\nu'-\lambda'}{r}\right)+\frac{1}{2}e^{-\nu}\left(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\right)=8\pi G T_2^2=8\pi G T_3^3,$$

$$-\frac{e^{-\lambda}}{r}\dot{\lambda}=8\pi G T_0^1, \quad \dot{\lambda}\equiv\frac{\partial\lambda}{\partial t}$$

The radial component of the equation $T^k_{i;k}=0$

$$\dot{T}_1^0+T_1^{1'}+\frac{1}{2}T_1^0(\dot{\nu}+\dot{\lambda})+\frac{\nu'}{2}(T_1^1-T_0^0)+\frac{2}{r}(T_1^1-T_2^2)=0.$$

Dynamical stability of incompressible anisotropic fluid stars

The energy–momentum tensor for a spherically symmetric anisotropic star with radial motions

$$T_i^k = (\varepsilon + p_t)u_i u^k - p_t \delta_i^k + (p_r - p_t)s_i s^k,$$

$u_i = dx_i/ds$ is the fluid four-velocity, s_i is the unit space-like vector with the properties

$$s^i u_i = 0, \; s^i s_i = -1.$$

For the motions in the radial direction

$$u^i = \left(\frac{e^{-\frac{\nu}{2}}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, \frac{v e^{-\frac{\nu}{2}}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, 0, 0\right), \quad v = \frac{dr}{dt}$$

$$s^i = \left(\frac{v e^{\frac{\lambda}{2}-\nu}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, \frac{e^{-\frac{\lambda}{2}}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, 0, 0\right)$$

The small radial oscillations

$$\varepsilon = \varepsilon^0 + \delta\varepsilon, \; p_r = p_r^0 + \delta p_r, \; p_t = p_t^0 + \delta p_t,$$
$$\nu = \nu^0 + \delta\nu, \; \lambda = \lambda^0 + \delta\lambda$$

Dynamical stability of incompressible anisotropic fluid stars

Let all perturbations $\delta\varepsilon, \delta p_r, \delta p_t, \delta\nu, \delta\lambda \sim e^{i\omega t}$.

Let us introduce a Lagrange displacement ψ: $v = \dot{\psi}$

After linearizing Einstein equations with respect to small perturbations, the equation for the frequencies of radial oscillations acquires the form

$$\omega^2 e^{\lambda^0-\nu^0}(\varepsilon^0 + p_r^0)\psi = \frac{2\psi}{r}p_r^{0'} - \frac{2\psi}{r}(\gamma(\nu^{0'} + \frac{\lambda^{0'}}{2} + \frac{2}{r}) + \frac{2}{r})(p_t^0 - p_r^0)$$

$$+8\pi G e^{\lambda^0} p_t^0(\varepsilon^0 + p_r^0)\psi - \gamma\frac{d}{dr}(\frac{2}{r}(p_t^0 - p_r^0)\psi) - \frac{\psi}{\varepsilon^0 + p_r^0}(p_r^{0'} - \frac{2}{r}(p_t^0 - p_r^0))^2$$

$$-\gamma e^{-(\nu^0+\frac{\lambda^0}{2})}\frac{d}{dr}(e^{\frac{3\nu^0+\lambda^0}{2}}\frac{p_r^0}{r^2}\frac{d}{dr}(r^2 e^{-\frac{\nu^0}{2}}\psi)) - \frac{2}{r}(\gamma p_r^0 \frac{e^{\frac{\nu^0}{2}}}{r^2}\frac{d}{dr}(r^2 e^{-\frac{\nu^0}{2}}\psi) + \delta p_t).$$

The boundary conditions

$$\psi(r=0) = 0, \delta p_r(r=R) = 0.$$

Dynamical stability of incompressible anisotropic fluid stars

In order to get the variational basis for finding ω, let us multiply both parts of the last equation on $r^2\psi\exp(\frac{\nu^0+\lambda^0}{2})$ and integrate over the range of *r.*

Omitting the upper indexes "0" as no longer necessary, for incompressible stars (γ→∞) one gets

$$\omega^2\int_0^R e^{\frac{3\lambda-\nu}{2}}(\varepsilon+p_r)r^2\psi^2 dr=\gamma\int_0^R e^{\frac{\lambda+3\nu}{2}}\frac{p_r}{r^2}\left(\frac{d}{dr}\left(r^2 e^{-\frac{\nu}{2}}\psi\right)\right)^2 dr$$

$$-\gamma\int_0^R e^{\frac{\lambda+\nu}{2}}r^2\psi\frac{d}{dr}\left(\frac{2}{r}(p_t-p_r)\psi\right)dr-2\gamma\int_0^R e^{\frac{\lambda+\nu}{2}}r\psi^2\left(\nu'+\frac{\lambda'}{2}+\frac{2}{r}\right)(p_t-p_r)$$

$$-2\gamma\int_0^R e^{\frac{\lambda}{2}+\nu}\psi\frac{p_r}{r}\frac{d}{dr}\left(r^2 e^{-\frac{\nu}{2}}\psi\right)dr.$$

The Lagrange displacement ψ should be chosen such that ω^2 is minimized.
A sufficient condition for the occurrence of the dynamical instability is vanishing of the r.h.s. of this equation for some trial form of the Lagrange displacement ψ satisfying the boundary conditions.

Dynamical stability of incompressible anisotropic fluid stars

After introducing the auxiliary function $\chi=e^{-\nu/2}\psi$, changing the integration variable $r=\alpha\xi$, $\alpha^2=q_0/4\pi G\rho_0$, substituting $p_r= q_0\rho_0\theta$, $\varepsilon= \rho_0$ and using expressions for Δ and metric functions at n=0:

$$\Delta(\xi)=\frac{p_{r0}\xi}{4}(1-\beta)\theta(\xi)\frac{\partial\nu}{\partial\xi},\quad e^{-\lambda}=1-\frac{2q_0\xi^2}{3},\qquad e^{\nu}=\frac{1-\frac{2GM}{R}}{(1+\beta q_0\theta)^{\frac{2}{\beta}}}$$

the variational equation reads

$$\frac{\omega^2}{\omega_0^2}\frac{1}{1-\frac{2GM}{R}}\int_0^{\xi_R}\frac{(1+q_0\theta)\xi^2\chi^2}{(1-\frac{2q_0\xi^2}{3})^{\frac{3}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{1}{\beta}}}=\gamma\int_0^{\xi_R}\frac{\theta(\frac{d}{d\xi}(\xi^2\chi))^2}{\xi^2(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}}$$

$$-\frac{\gamma(1-\beta)}{2}\int_0^{\xi_R}\frac{\xi^2\chi}{(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}}\frac{d}{d\xi}\left(\frac{\nu'\chi\theta}{(1+\beta q_0\theta)^{\frac{1}{\beta}}}\right)\frac{d\xi}{(1+\beta q_0\theta)^{\frac{2}{\beta}}}$$

$$-\frac{\gamma(1-\beta)}{2}\int_0^{\xi_R}\frac{\xi^2\chi^2\nu'\theta(\nu'+\frac{\lambda'}{2}+\frac{2}{\xi})}{(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}}-2\gamma\int_0^{\xi_R}\frac{\chi\theta\frac{d}{d\xi}(\xi^2\chi)}{\xi(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}},$$

$$\omega_0^2=4\pi\rho_0 G$$

Dynamical stability of incompressible anisotropic fluid stars

The trial functions

$$\chi_1 = e^{-\frac{\nu}{2}}\xi, \quad \chi_2 = \sqrt{\xi}.$$

The critical values of the parameter q_0 for the appearance of the dynamical instability of an incompressible anisotropic fluid star at different values of the parameter β

β	q_{0c} evaluated with the trial function	
	$\chi_1=e^{-\nu/2}\xi$	$\chi_2=\sqrt{\xi}$
0.1	1.391	-
0.3	1.796	-
0.5	2.526	-
0.7	4.210	-
0.9	11.646	-

The local pressure anisotropy with $\Delta=p_t-p_r>0$ can make the incompressible fluid stars dynamically unstable.

Thank you for attention