Wormholes with fluid sources: A no-go theorem and new examples

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3rd international conference on particle physics and astrophysics
Moscow, 3 October 2017

Motivation and plan

- Wormholes: why GR, why spherical symmetry?
- Fluid formalism in spherical symmetry: most general problem statement
- Isotropic matter: no-go theorem. No asympt. flat/AdS wormholes
- Isotropic matter: examples of asympt. de Sitter wormholes
  Symmetric and asymmetric wormholes w.r.t. the throat
- Anisotropic matter: asympt flat solutions with $R = 0$.
  Symmetric and asymmetric wormholes w.r.t. the throat
  Relation to brane worlds.
- Mathematical curiosity: intersections of integral curves

The Einstein equations can be written in two equivalent forms

$$G_\mu^\nu \equiv R_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu R = -T_\mu^\nu, \quad \text{or} \quad R_\mu^\nu = -(T_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu T_\alpha^\alpha), \tag{1}$$

T_μ^ν = stress-energy tensor (SET) of matter.

Metric :
$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)(d\theta^2. + \sin^2\theta d\varphi^2). \tag{2}$$

The most general SET compatible with (2):

$$T_\mu^\nu = \mathrm{diag}(\rho,\ -p_r,\ -p_T,\ -p_T), \tag{3}$$

ρ = energy density, p_r = radial pressure, p_T = tangential pressure.

Wormhole: $r(x)$ has a regular minimum (say, at $x = x_0$), called a throat, and reaches values much larger than $r(x_0)$ on both sides; $A > 0$ at least near x_0.

The difference $\binom{t}{t} - \binom{x}{x}$ of the Einstein equations reads

$$2A\,r''/r = -(T_t^t - T_x^x) \equiv -(\rho + p_r), \tag{4}$$

At a minimum of $r(x)$ we have $r' = 0, \quad r'' > 0$. Hence (4) is **NEC violation**.

Three nontrivial components in the Einstein equations:

$$G_t^t = r^{-2}\left[-1 + A(2rr'' + r'^2) + A'rr'\right] = -\rho, \qquad (5)$$
$$G_x^x = r^{-2}\left[-1 + A'rr' + Ar'^2\right] = p_r, \qquad (6)$$
$$G_\theta^\theta = G_\phi^\phi = r^{-2}\left[Ar'' + \tfrac{1}{2}rA'' + A'r'\right] = p_T, \qquad (7)$$

If we require $p_r = p_T$ (isotropic matter), we have
$r^2A'' + 2Arr'' - 2Ar'^2 + 2 = 0$, or, substituting $A(x) = D(x)/r^2(x)$,

$$D'' - 4D'r'/r + 4Dr'^2/r^2 + 2 = 0. \qquad (8)$$

If at some $x = x_0$, $D \geq 0$, $D' = 0$, then by (8), $D'' \leq -2$, so it is a maximum. However, at a flat asymptotic $A \to 1$, $D \sim r^2 \to \infty$; at an AdS asymptotic $A \sim r^2$, $D \sim r^4 \to \infty$. If there are two such regions, there is a minimum of $D(x)$ between them — but it is impossible by (8).

Theorem. *A static, spherically symmetric traversable wormhole with $r \to \infty$ and $A(x)r^2(x) \to \infty$ on both sides of the throat cannot be supported by any isotropic matter source with $p_r = p_T$.*

No clear reason to assume a particular equation of state
$\Rightarrow$ we instead specify a suitable metric function $r(x)$:

$$r(x) = \sqrt{a^2 + x^2}, \qquad a = \mathrm{const} > 0. \tag{9}$$

For a numerical study, we put $a = 1$ (the length scale remains arbitrary). Assuming $p_r = p_T$, we use Eq. (8) to find $A(x)$. With (9) it takes the form

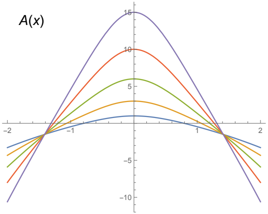
$$(1 + x^2)^2 A'' + 2(1 - x^2)A + 2(1 + x^2) = 0. \tag{10}$$

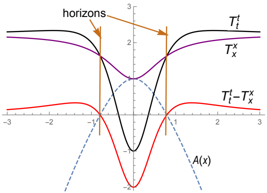
After solving it (only numerically!), the metric is known completely. At large x, the solution of (10) is

$$A(x) = 1 + c_1 x^2 + c_2/x, \qquad c_{1,2} = \mathrm{const}.$$

compatible with both flat and (A)dS behavior.

Isotropic matter: symmetric dS-dS wormholes

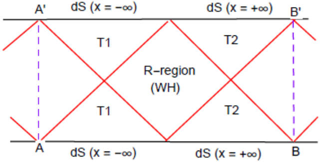




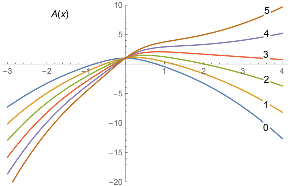
Left: Solutions $A(x)$ of Eq. (10) for a symmetric dS–dS wormhole, with $A(0) = 1, 3, 6, 10, 15$ (bottom-up along the ordinate axis) and $A'(0) = 0$.

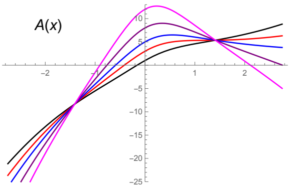
Right: The metric function $A(x)$ and the SET components for a symmetric dS–dS wormhole according to Eq. (10) with $A(0) = 1$ and $A'(0) = 0$.

A point of interest: all curves $A(x)$ intersect at two symmetric points: $x \approx \pm 1.4109$, $A(x) \approx -1.4953$.



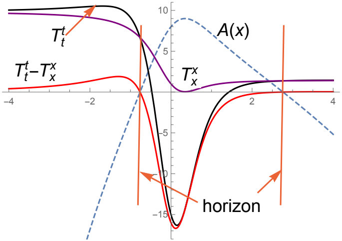
Carter-Penrose diagram of a dS-dS wormhole. Lines AA' and BB': possible identification, it means that the wormhole connects regions of the same de Sitter universe





Left: The function $A(x)$ for $A(0) = 1$ and $A'(0) = 0, 1, 2, 3, 4, 5$ (written on the corresponding curves). Two upper curves $\to$ AdS as $x \to +\infty$.

Right: The function $A(x)$ for the same slope at the throat, $A'(0) = 6$ and $A(0) = 1, 3, 5, 8, 12$ (bottom-up on the ordinate axis and conversely at large $|x|$). The curves intersect at the same x as for symmetric models.



The function $A(x)$ and the SET components for the solution with $A(0) = 8$ and $A'(0) = 6$, having two de Sitter asymptotics.with different curvature values.

Now let us abandon the source isotropy assumption and try to obtain new models of twice asymptotically flat geometries. As before, there is no clear reason to assume a particular form of the equations of state (which are now different for p_r and p_T). Instead, we again take $r(x)$ in the form $r = \sqrt{1+x^2}$.

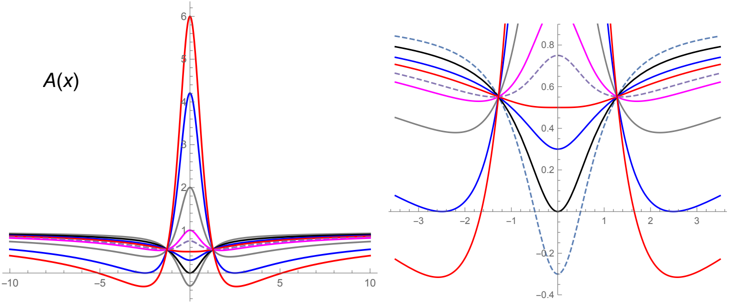
In addition, we assume the Ricci scalar $R = 0$. Hence it is possible to interpret the results as vacuum solutions in an Randall-Sundrum-2-like brane world. For our metric (2) we have

$$R = \frac{2}{r^2} - A'' - 4A'\frac{r'}{r} - 4A\frac{r''}{r} - 2A\frac{r'^2}{r^2}. \qquad (11)$$

For our choice of $r(x)$, the equation $R = 0$ takes the form

$$A'' + \frac{4x}{1+x^2}A' + \frac{2(2+x^2)}{(1+x^2)^2}A = \frac{2}{1+x^2}. \qquad (12)$$

At large $|x|$ the asymptotic form of its solution is $A = 1 + C_1/x + C_2/x^2$, $C_{1,2} = \mathrm{const}$, $\Rightarrow$ Schwarzschild-like asymptotic flatness.

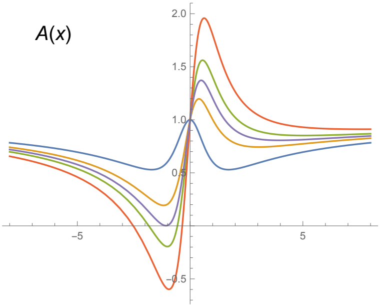


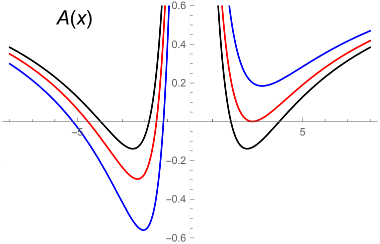
Symmetric solutions to Eq. (12) with $A'(0) = 0$ and $A(0) = -0.3,\ 0,\ 0.5,\ 0.75,\ 1,\ 2,\ 4.205,\ 6$ (bottom-up at small x, conversely at large $|x|$).

Left — a general picture,

Right — its part of interest enlarged.

There are **wormholes and regular black holes with two or four horizons**.





Asymmetric solutions to Eq. (12).
Left: $A(0) = 1$ and $A'(0) = 0, 1, 1.5, 2, 3$ (upside-down for $x < 0$ and bottom-up for $x > 0$).
Right: $A(0) = 5$ and $A'(0) = 0, 0.8, 2$ (upside-down for $x < 0$ and bottom-up for $x > 0$). The peaks near $x = 0$ are similar to the symmetric case.

The number of horizons if from zero to four.

Conclusions

- A no-go theorem showing that it is impossible to obtain static asymptotically flat or AdS wormholes without horizons, supported by isotropic matter.
- With isotropic matter: a family of wormholes which connect two de Sitter worlds with the same or different curvature. They can link distant regions of, e.g., the same inflationary universe making them causally connected. (Unlike other models where the throat cosmologically expands, here its radius is constant.)
  Also dS-M and dS-AdS configurations — black universes.
- With anisotropic matter: new (numerical) asymptotically flat models of wormholes and regular BHs, both $\mathbb{Z}_2$-symmetric and asymmetric, with up to 4 Killing horizons satisfying $R = 0 \quad \Rightarrow$ vacuum solutions in a brane world.
- Intersections of integral curves of linear ODE, observed for both $R_x^x = R_\theta^\theta$ and $R = 0$. A general property of such equations.

Consider a general linear real-valued 2nd order ODE for $y(x)$:

$$A(x)y'' + B(x)y' + C(x)y = F(x), \tag{A1}$$

with the initial conditions $y(x_0) = a, \quad y'(x_0) = b.$

Suppose we know $y_1(x)$, $y_2(x)$ — linearly independent solutions of the homogeneous equation, and $y_3(x)$ — a special solution to the inhomog. eq. Then the general solution to (A1) may be written as

$$y(x,a,b) = \frac{y_1(x)}{W_0}[(a-y_{30})y'_{20} - (b-y'_{30})y_{20}] - \frac{y_2(x)}{W_0}[(a-y_{30})y'_{10} - (b-y'_{30})y_{10}]$$
$$+y_3(x); \qquad W_0 = \begin{vmatrix} y_{10} & y_{20} \\ y'_{10} & y'_{20} \end{vmatrix} \tag{A2}$$

with the constants $y_{i0} = y_i(x_0), \quad y'_{i0} = y'_i(x_0), \quad i = 1,2,3.$

Question: If we fix $b = y'(x_0)$ and vary $a = y(x_0)$, will the curves (A2) intersect?
Answer: Yes, at such $x = x_*$ that $\partial y(x,a,b)/\partial a = 0 \quad \Rightarrow$ the equation

$$y_1(x_*)y'_{20} = y_2(x_*)y'_{10}. \tag{A3}$$

It is insensitive to the choice of $y_1(x)$ and $y_2(x)$, and independent of b and $F(x)$. It can have any number of solutions, from zero to infinity.

THANK YOU!