Rotating cylindrical wormholes: a no-go theorem

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ICPPA 2015
International Conference on Particle Physics and Astrophysics
October 5-10, 2015
Moscow, Russia, Hotel Milan.

Plan

<u>Basic problem:</u> is it possible to build a WH in GR without exotic matter?

(It is possible in some other theories of gravity; but GR is very well confirmed on the macro level, so the question is important)

1. Cylindrical wormholes: different versions of throat definition
   Flat or string asymptotics.
2. Static cylindrical wormholes.
   A no-go theorem: $\rho < 0$ for twice as. regular wormholes
3. Rotation. Structure of the equations.
   Example: vacuum and scalar-vacuum solutions.
   Bad asymptotic behavior.
4. Thin shells: attempt to construct a realistic wormhole by joining the throat region with flat space regions.
   - **A no-go theorem**

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Wormholes (spherical vs. cylindrical)

Static, spherically symmetric space-times:

$$ds^2 = A(u)dt^2 - B(u)du^2 - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2)$$

A wormhole throat: a regular minimum of ***r(u):*** $r = r_{throat}$

A wormhole: a regular configuration where ***A(u)* >0** and ***B(u)* >0** everywhere (*no horizons*), and, far from the throat, on its both sides, $r \gg r_{throat}$ **As. flatness – the most interesting case.**

The Universe may also contain structures **infinitely extended** along a certain direction, like cosmic strings. While starlike structures are, in the simplest case, described by spherical symmetry, the simplest stringlike configurations are **cylindrically symmetric:**

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,$$

Static, spherically symmetric wormholes: basic facts

$$ds^2 = A(u)dt^2 - B(u)du^2 - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2)$$

At the throat: $r' = 0$, $r'' > 0$ =>

for matter of general form compatible with the symmetry, $T_\mu^\nu = \mathrm{diag}(\rho, -p_r, -p_\perp, -p_\perp)$,

these conditions lead to $\rho + p_r < 0, \qquad p_r < 0.$

("Exotic" matter, violation of the Null Energy Condition. But no restriction on sign ρ)

This result is valid for any static throats with sph. topology (Hochberg, Visser, 1997)

Flat asymptotic: at large r approx.Schwarzschild, with mass m.

Mass function: $m(r) = 4\pi G \int_{r_0}^{r} \rho r^2 dr,$ r_0 = integration constant.

At the throat, $2m(r) = r$. Integrating from $r_0 = r_{th}$, we obtain $r_{\rm th} = 2m - \varkappa \int_{r_{\rm th}}^{\infty} \rho r^2 dr,$

This means: if $\rho > 0$, then $r_{th} < 2m = r$ (Schwarzschild).

a wormhole with a throat of a few meters will have **huge gravity** of, say, Jupiter !!

To avoid that, negative densities are necessary.

Wormholes (spherical vs. cylindrical)

Static, cylindrically symmetric space-times: their general metric can be taken in the form

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,$$

where u = arbitrary admissible cylindrical radial coordinate,
z = longitudinal, ϕ = angular coordinate.

Definition:

A wormhole throat: a regular minimum of $r(u) \equiv e^{\beta}$: $r = r_{th}$

A wormhole: a regular configuration where, far from the throat, on its both sides, $r \gg r_{th}$.\

Alternative definition:

the same, but using, instead of $r(u)$ [the circular radius],
$a(u) \equiv \exp(\beta+\xi)$ [the area function].

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,$$

Boundary conditions for cylindrical wormholes

Consider the most natural situation that the wormhole is observed as a stringlike source of gravity from an otherwise very weakly curved or even flat environment.

We require: there is a spatial infinity, i.e., at some $u = u_\infty$, $r \equiv e^\beta \to \infty$, the metric is either flat or corresponds to the gravitational field of a cosmic string.

This means: (1) $\gamma \to \text{const}, \quad \xi \to \text{const} \quad \text{as } u \to u_\infty.$

(2) at large *r*, $|\beta'| e^{\beta - \alpha} \to 1 - \mu, \quad \mu = \text{const} < 1 \quad \text{as } u \to u_\infty$

(the parameter μ is an angular defect). A ***flat asymptotic:*** $\mu = 0$.

We say ***"a regular asymptotic"*** in the sense ***"a flat or string asymptotic."***

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,$$

Einstein equations: $G_\mu^\nu = -\varkappa T_\mu^\nu, \qquad \varkappa = 8\pi G,$

or $R_\mu^\nu = -\varkappa \tilde{T}_\mu^\nu, \qquad \tilde{T}_\mu^\nu = T_\mu^\nu - \frac{1}{2}\delta_\mu^\nu T_\alpha^\alpha$

We have

$$R_0^0 = -e^{-2\alpha}[\gamma'' + \gamma'(\gamma' - \alpha' + \beta' + \xi')],$$

$$R_1^1 = -e^{-2\alpha}[\gamma'' + \xi'' + \beta'' + \gamma'^2 + \xi'^2 + \beta'^2 - \alpha'(\gamma' + \xi' + \beta')],$$

$$R_2^2 = -e^{-2\alpha}[\xi'' + \xi'(\gamma' - \alpha' + \beta' + \xi')],$$

$$R_3^3 = -e^{-2\alpha}[\beta'' + \beta'(\gamma' - \alpha' + \beta' + \xi')],$$

$$G_1^1 = e^{-2\alpha}(\gamma'\xi' + \beta'\gamma' + \beta'\xi').$$

The most general stress-energy tensor: $T_\mu^\nu = \mathrm{diag}(\rho, -p_r, -p_z, -p_\phi),$

where ρ = energy density,

p_i = pressures of any physical origin in the respective directions.

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,$$

Conditions on the throat

Harmonic radial coordinate is used: $\alpha = \beta + \gamma + \xi$

1. **At a minimum of** ***r (u)***, due to **β'=0 and β'' > 0**, we have $R_3^3 < 0$,
   and from the corresponding component of the Einstein eqs it follows that

$$T_0^0 + T_1^1 + T_2^2 - T_3^3 = \rho - p_r - p_z + p_\phi < 0.$$

If $T_2^2 = T_3^3$, that is, $p_z = p_\phi$, in particular, for Pascal isotropic fluids we obtain $p_r > \rho$, violation of ***Dominant Energy Condition*** (if, as usual, ρ > 0).

In the general case of anisotropic pressures, none of the standard energy conditions are necessarily violated.

2. However, **if the throat is defined through** ***the area function*** ***a(u)*** **≡** $e^{\beta+\xi}$,
   we have there β' + ξ' = 0, β'' + ξ'' > 0, whence $R_2^2 + R_3^3 < 0 \Rightarrow T_0^0 + T_1^1 = \rho - p_r < 0.$

   In addition, substituting β' + ξ' = 0 into the Einstein equation $G_1^1 = -\varkappa T_1^1$, we find

   $-T_1^1 = p_r \leq 0.$ Combining these two conditions, we see that

$$\rho < p_r \leq 0$$

on the throat, i.e., there is necessarily a region with negative energy density !!!

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2,$$

Asymptotic regularity and a no-go theorem

At a regular asymptotic, both *r (u)* and *a(u)* tend to infinity. **If there are two such asymptotics, both functions have minima** at some finite *u*, i.e., there occur both **a throat as a minimum of** *r (u)* and **a throat as a minimum of** *a(u)* (they do not necessarily coincide if there is no mirror symmetry).

This leads to the following result (**no-go theorem**) :

In general relativity, any static, cylindrically symmetric wormhole with two regular asymptotics contains a region where the energy density is negative.

Another formulation:

In general relativity, a static, cylindrically symmetric, twice asymptotically regular wormhole cannot exist if the energy density $\rho = T_0^0$ ***is everywhere nonnegative.***

CONCLUSION on static cyl. wormholes

- Cyl. wormhole geometries can exist without WEC or NEC violation
- A number of explicit examples of cyl. wormholes with non-phantom matter are known
- Main difficulty (*as always with cyl. symmetry*): obtaining flat or string (regular) asymptotics.
- It is necessary to have **negative density** to obtain twice asympt. regular wormholes.

The latter problem is still more important if we try to apply cylindrically symmetric solutions as an approximate description of toroidal systems. This approximation must work well if a torus containing matter and significant curvature is thin, like a circular string.

Rotation

A vortex gravitational field in terms of tetrads: $\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} e_{a\nu} e^a_{\rho;\sigma},$

Cylindrical symmetry: metric

$$ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}\, d\varphi]^2 - e^{2\alpha(u)}du^2 - e^{2\mu(u)}dz^2 - e^{2\beta(u)}d\varphi^2,$$

The vortex gravitational field is characterized by

$$\omega = \tfrac{1}{2}(Ee^{-2\gamma})' e^{\gamma-\beta-\alpha}. \qquad \omega = \sqrt{\omega_\alpha \omega^\alpha}$$

Ricci tensor component (~ flux) $R^3_0 \sim (\omega e^{2\gamma+\mu})' = 0$

=> in the comoving reference frame

$$\omega = \omega_0 e^{-\mu-2\gamma}, \qquad \omega_0 = \text{const.}$$

$$ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}\, d\varphi]^2 - e^{2\alpha(u)}du^2 - e^{2\mu(u)}dz^2 - e^{2\beta(u)}d\varphi^2,$$

In the comoving reference frame (arbitrary gauge):

$$R^1_1 = -\,e^{-2\alpha}[\beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'^2 - \alpha'(\beta' + \gamma' + \mu')] + 2\omega^2;$$
$$R^2_2 = \Box_1 \mu;$$
$$R^3_3 = \Box_1 \beta + 2\omega^2$$
$$R^4_4 = \Box_1 \gamma - 2\omega^2$$

where $\Box_1 f = -g^{-1/2}[\sqrt{g}g^{11} f']' = -\,e^{-2\alpha}[f'' + f'(\beta' + \gamma' + \mu' - \alpha')].$

so that the Einstein tensor is

time

$$G^\nu_\mu = {}_sG^\nu_\mu + {}_\omega G^\nu_\mu, \qquad {}_\omega G^\nu_\mu = \omega^2 \,\mathrm{diag}(-3,\ 1,\ -1,\ 1).$$

$\underset{s}{G^\nu_\mu}$ and $\underset{\omega}{G^\nu_\mu}$ (each separately) satisfy the "conservation law"

$\nabla_\alpha G^\alpha_\mu = 0$ with respect to the static metric with $E \equiv 0$

The rotational part of the Einstein tensor behaves in the Einstein equations as an additional SET with very exotic properties: the energy density is $-3\omega^2/\varkappa < 0$

$$ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}\, d\varphi]^2 - e^{2\alpha(u)}du^2 - e^{2\mu(u)}dz^2 - e^{2\beta(u)}d\varphi^2,$$

Definitions: the same as in the static case.

r-throat: a regular minimum of ***r*(*u*) = exp (β)**
r-wormhole: a regular configuration with ***r* >> *r*$_{min}$** on both sides

a-throat: a regular minimum of ***a*(*u*) = exp (β+μ)**
a-wormhole: a regular configuration with ***a* >> *a*$_{min}$** on both sides

Throat conditions: if $T_1^1 = -p_r, \quad T_2^2 = -p_z, \quad T_3^3 = -p_\varphi, \quad T_4^4 = \rho,$

then on an r-throat $\rho - p_r - p_z + p_\varphi - 2\omega^2/\varkappa < 0.$

and on an a-throat $T_1^1 + T_4^4 = \rho - p_r < 2\omega^2/\varkappa. \qquad -T_1^1 = p_r \le \omega^2/\varkappa.$

Much easier to obtain wormholes than with ω = 0.
Main problem: bad asymptotic behavior,
e.g., we do not have **ω → 0** where **γ, μ → const** since

$$\omega = \omega_0\, e^{-\mu-2\gamma}, \qquad \omega_0 = \text{const}.$$

Example:

$$ds^2 = e^{2\alpha}du^2 + e^{2\mu}dz^2 + e^{2\beta}d\varphi^2 - e^{2\gamma}(dt - E\,e^{-2\gamma}\,d\varphi)^2$$

vacuum or a massless scalar field,

$$L_s = -\frac{1}{2}\varepsilon(\partial\phi)^2 - V(\phi)$$

ε = +1 normal,
ε = -1 phantom

Harmonic radial coordinate: $\alpha = \beta + \gamma + \mu$

$$\begin{aligned} R_2^2 = 0 \quad &\Rightarrow \quad \mu'' = 0, \\ R_3^3 = 0 \quad &\Rightarrow \quad \beta'' - 2\omega^2 e^{2\alpha} = 0, \\ R_4^4 = 0 \quad &\Rightarrow \quad \gamma'' + 2\omega^2 e^{2\alpha} = 0, \end{aligned}$$

⇒

$$\begin{aligned} &\mu = -mu \qquad [\text{with a certain choice of } z \text{ scale}], \\ &\beta + \gamma = 2hu \quad [\text{with a certain choice of } t \text{ scale}], \\ &\beta'' - \gamma'' = 4\omega_0^2\,e^{2\beta - 2\gamma}. \end{aligned}$$

Solution:

$$\phi = Cu$$

$$\omega = \frac{e^{mu-2hu}}{2s(k,u)},$$

$$\begin{aligned} e^{2\beta} &= \frac{e^{2hu}}{2\omega_0 s(k,u)}, \\ e^{2\gamma} &= 2\omega_0 s(k,u)\,e^{2hu}, \\ e^{2\mu} &= e^{-2mu}, \end{aligned}$$

$$E = e^{2hu} s(k,u) \int du\, \frac{e^{2hu}}{s(k,u)}.$$

$$k^2\,\mathrm{sign}\,k = 4(h^2 - 2hm) - 2\varkappa\varepsilon C^2.$$

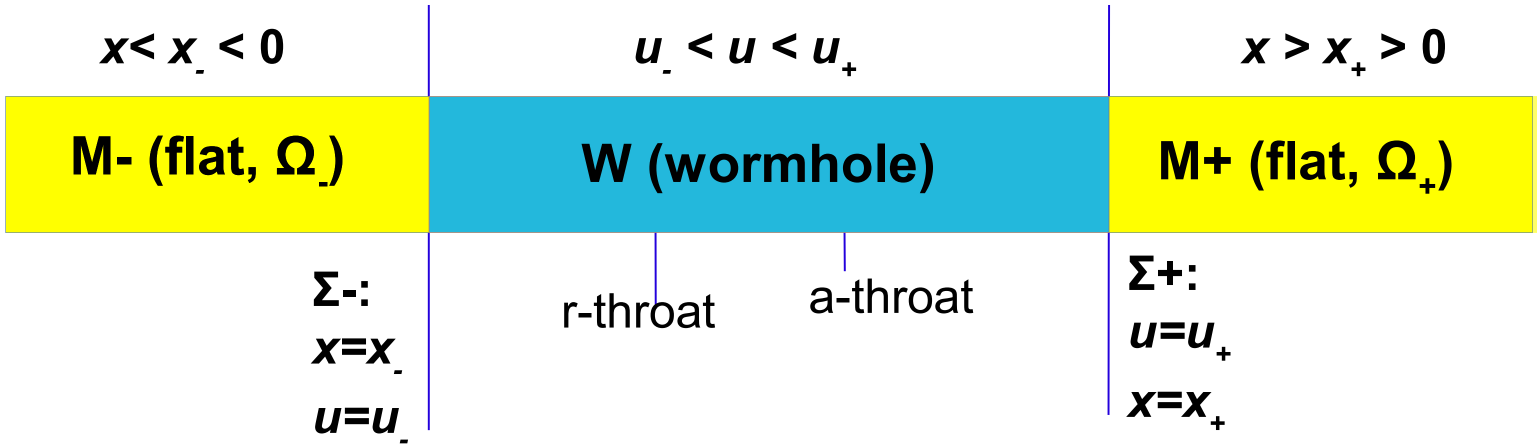
$$s(k,u) = \begin{cases} k^{-1}\sinh ku, & k > 0, \quad u \in \mathbb{R}_+; \\ u, & k = 0, \quad u \in \mathbb{R}_+; \\ k^{-1}\sin ku, & k < 0, \quad 0 < u < \pi/|k|. \end{cases}$$

Parameters: ω_0, *C, h, m, k.* (All inessential constants have been absorbed.)

Wormhole solutions: **all solutions with** ***k*** **< 0, and many with** ***k*** **≥ 0.**

Asymptotic behavior: **in all cases, either exp(γ) → 0, or exp(γ) → ∞.**

Trying to build a wormhole model with two flat asymptotics



Metrics in M+ and M-: Minkowski $ds^2_{\mathrm{M}} = dt^2 - dx^2 - dz^2 - x^2(d\varphi + \Omega dt)^2$

$$e^{2\beta} = \frac{x^2}{1 - \Omega^2_\pm x^2}; \qquad e^{\mu} = e^{\mu_\pm}, \qquad e^{2\gamma} = e^{2\gamma_\pm}(1 - \Omega^2_\pm x^2);$$

$$e^{\alpha} = 1, \qquad \omega = \frac{\Omega_\pm}{1 - \Omega^2_\pm x^2}, \qquad E = \Omega_\pm x^2 e^{\gamma_\pm},$$

(with arbitrary scales along the *z* and *t* axes)

Matching: $[\beta] = 0, \qquad [\mu] = 0, \qquad [\gamma] = 0, \qquad [E] = 0,$

Meaning: we identify the cylinders Σ in the external (Minkowski) and internal (wormhole) space-times by adjusting the parameters of both

Next step:

find the surface stress-energy tensors on Σ+ and Σ-

This is done in Darmois-Israel formalism in terms of the extrinsic curvature:

$$S_a^b = -\frac{1}{8\pi}[\tilde{K}_a^b], \qquad \tilde{K}_a^b := K_a^b - \delta_a^b K, \qquad K = K_a^a, \qquad \text{[f] := f(+) - f(-).}$$

With natural parametrization of **Σ+** and **Σ-** $K_{ab} = -\,\mathrm{e}^{\alpha(u)}\Gamma^1_{ab} = \frac{1}{2}\mathrm{e}^{-\alpha(u)}\frac{\partial g_{ab}}{\partial x^1}.$

⇨

$$\tilde{K}_2^2 = -\,\mathrm{e}^{-\alpha}(\beta' + \gamma'),$$
$$\tilde{K}_3^3 = -\,\mathrm{e}^{-\alpha}(\mu' + \gamma') - E\omega\,\mathrm{e}^{-\beta-\gamma},$$
$$\tilde{K}_4^4 = -\,\mathrm{e}^{-\alpha}(\beta' + \mu') + E\omega\,\mathrm{e}^{-\beta-\gamma},$$
$$\tilde{K}_4^3 = \omega\,\mathrm{e}^{\gamma-\beta}.$$

$-S_4^4$ is the surface density while $S_2^2 = p_z$ and $S_3^3 = p_\varphi$ are pressures in the respective directions.

(**No need** to adjust coordinates in different regions since all relevant quantities are reparametrization-independent.)

Can both surface stress-energy tensors be physically plausible and non-exotic at some values of the system parameters?

Criterion: the WEC (including the NEC)

$$\frac{S_{00}}{g_{00}} = \sigma \geq 0, \qquad [\tilde{K}_{00}/g_{00}] \leq 0,$$

$$S_{ab}\xi^a\xi^b \geq 0, \qquad [K_{ab}\xi^a\xi^b] \leq 0.$$

We choose the null vectors in z and φ directions:

$$\xi^a_{(1)} = (\mathrm{e}^{-\gamma},\ \mathrm{e}^{-\mu},\ 0),$$

$$\xi^a_{(2)} = (\mathrm{e}^{-\gamma} + E\mathrm{e}^{-\beta-2\gamma},\ 0,\ \mathrm{e}^{-\beta}),$$

Then our requirements read:

$$[\mathrm{e}^{-\alpha}(\beta' + \mu')] \leq 0,$$

$$[\mathrm{e}^{-\alpha}(\mu' - \gamma')] \leq 0,$$

$$[\mathrm{e}^{-\alpha}(\beta' - \gamma') + 2\omega] \leq 0,$$

This was a general consideration. Now consider the Einstein equations for our metric

$$ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}\, d\varphi]^2 - e^{2\alpha(u)} du^2 - e^{2\mu(u)} dz^2 - e^{2\beta(u)} d\varphi^2$$

One of them reads $$\beta'' - \gamma'' - 4\omega_0^2 e^{2\beta-2\gamma} = \varkappa e^{2\alpha}(T_t^t - T_\varphi^\varphi).$$

Therefore, if the SET of matter satises the condition $T_t^t = T_\varphi^\varphi$,

it is easily integrated giving

$$e^{\beta-\gamma} = \frac{1}{2|\omega_0| s(k,u)}, \qquad k = \text{const},$$

$$s(k,u) = \begin{cases} k^{-1}\sinh ku, & k > 0, \quad u \in \mathbb{R}_+; \\ u, & k = 0, \quad u \in \mathbb{R}_+; \\ k^{-1}\sin ku, & k < 0, \quad 0 < u < \pi/|k|. \end{cases}$$

At ***k* < 0** **there are wormhole metrics** (provided other metric functions are regular,
And **there can be wormhole solutions** with ***k* > 0** (as the vacuum example shows) .

The condition $T_t^t = T_\varphi^\varphi$ holds for a large class of matter Lagrangians

In our metric, for example, $$L = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi) - P(\phi)F^{\mu\nu}F_{\mu\nu},$$

with an arbitrary scalar field potential $V(\phi)$ and an arbitrary function $P(\phi)$ characterizing the scalar-electromagnetic interaction, assuming that $\phi = \phi(u)$ and that the Maxwell tensor $F_{\mu\nu}$ describes a stationary azimuthal magnetic field ($F_{21} = -F_{12} \neq 0$) or its electric analogue.

Now, our requirement $[e^{-\alpha}(\beta' - \gamma') + 2\omega] \le 0.$ reads

On Σ- (x < 0)

$$e^{-\alpha(u_-)} \frac{(-s' + \operatorname{sign}\omega_0)}{s} + \frac{(1+\Omega_- x)^2}{|x|(1-\Omega_-^2 x^2)} \le 0,$$

On Σ+ (x > 0)

$$e^{-\alpha(u_+)} \frac{(s' - \operatorname{sign}\omega_0)}{s} + \frac{(1+\Omega_+ x)^2}{x(1-\Omega_+^2 x^2)} \le 0.$$

At $\omega_0 > 0,$ for Σ- => $s' > 1$, it is only possible with $k > 0$.
Same for Σ+ => $s' < 1$, it is only possible with $k < 0$ since

$$s'(k,u) = \{\cosh k,\ 1,\ \cos|k|u\} \text{ for } k>0,\ k=0, \text{ and } k<0, \text{ respectively}$$

But there is a single solution in **V**, with fixed parameters, including *k*
=> our requirements lead to a contradiction.
(It is true for arbitrarily rotating shells on Σ+ or Σ-).

If $\omega_0 < 0,$ the condition on Σ+ cannot be fulfilled at any *k*

=>The NEC is inevitably violated on Σ+ or Σ- (or on both)
(NO-GO THEOREM)

Conclusion on rotating cylindrical wormholes in GR

Stationary cylindrical configurations:

The vortex grav. field is singled out and behaves like matter with exotic properties. Some exact solutions are found.

Is it possible to have WH solutions without exotic matter?

YES (vortex gravitational field instead of WEC violating matter)

Can they have two flat (or string) asymptotic regions?

NO if we consider pure solutions with rotation.

Probably NO with Minkowski regions and rotating thin shells.

(Though, we can hope to solve the problem with other kinds of matter instead of those ruled out by the no-go theorem.)

This story did not have a happy end, but it is not an end anyway.

THANK YOU!