Vacuum polarization of a quantized scalar field in the thermal state in a long throat

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- Definitions
- Wormholes created by classical fields
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- Model of the wormhole with an infinitely short throat in frame of semiclassical theory of gravity
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- Model of the wormhole with a long throat in frame of semiclassical theory of gravity
- Conclusion

Wormholes are topological handles connecting separated regions of a single universe or two different spacetimes

The metric of a static spherically symmetric wormhole in the curvature coordinates:

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right), \tag{1}$$

Embedding diagram:

- The section $t = const, \theta = \pi/2$:

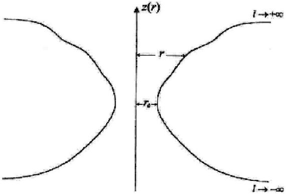
$$ds^2 = \frac{dr^2}{1 - b(r)/r} + r^2 d\varphi^2$$

embedded in three-dimensional Euclidean space

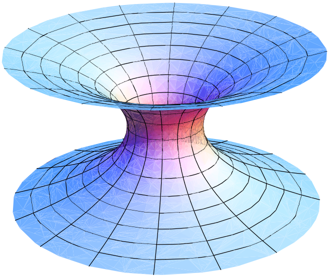
$$ds^2 = dz^2 + dr^2 + r^2 d\varphi^2.$$

- Embedding conditions:

$$z = z(r), \quad \frac{dz}{dr} = \pm \left(\frac{r}{b(r)} - 1 \right)^{-1/2}.$$



- Embedding diagram:



$G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu} \qquad (\hbar = c = G = 1)$

At the throat of a wormhole

$\varepsilon + p < 0$ – null energy condition is violated

Alternative theories of gravity

- scalar-tensor theories;
- theory of gravity with torsion;
- asymmetric ($g_{\mu\nu} \neq g_{\nu\mu}$) theory of gravity;
- theory of gravity with higher derivatives;
- etc.

Wormholes created by dark energy

Sushkov (2005), Lobo (2005)

Classical scalar field:

- Classical action:

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{8\pi} - (\nabla\phi)^2 - \xi R\phi^2 - 2V(\phi)\right\} \tag{2}$$

- Stress-energy tensor:

$$\begin{aligned} T_{\mu\nu} = & \nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 - g_{\mu\nu}V(\phi) \\ & + \xi\left[G_{\mu\nu}\phi^2 - 2\nabla_\mu(\phi\nabla_\nu\phi) + 2g_{\mu\nu}\nabla^\lambda(\phi\nabla_\lambda\phi)\right] \end{aligned} \tag{3}$$

- Null energy condition:

$$8\pi T_{\mu\nu}K^\mu K^\nu = G_{\mu\nu}K^\mu K^\nu = \frac{8\pi}{1 - 8\pi\xi\phi^2}\left[\phi'^2 - \xi(\phi^2)''\right] \geq 0 \tag{4}$$

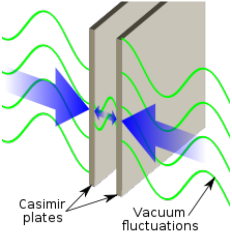
K^μ – arbitrary null vector vector

may be violated for some values of ξ.

Explicit solutions that describe the wormholes of this type were obtained by Bronnikov (1973), Barcelo and Visser (1999, 2000), Sushkov and Kim (2002).

Morris M. and Thorne K. (Am. J. Phys., 1988)

- Casimir effect:



$$T_{\mu\nu} = \frac{\pi^2}{720} \frac{\hbar}{a^4} diag(-1, 1, 1, -3), \qquad (5)$$

$\varepsilon = -\dfrac{\pi^2 \hbar}{720 a^4} < 0$ – energy density is negative

$\varepsilon + p = -\dfrac{\pi^2 \hbar}{180 a^4} < 0$ – null energy condition is violated

- Squeezed states in quantum electrodynamics

Semiclassical theory of gravitation

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle_{ren}, \quad (6)$$

$\langle T_{\mu\nu}\rangle_{ren}$ – the vacuum expectation value of the stress-energy tensor of quantized matter fields.

$\langle T_{\mu\nu}\rangle_{ren}$ is determined by the geometrical and topological properties of the space-time, as well as the choice of a quantum state

The characteristic scale of the components G^{μ}_{ν} on the left-hand side of equations (6) is $1/l^2$, on the right-hand side - $l_{\rm Pl}{}^2/(l^4)$

$$\Rightarrow \quad l \simeq l_{\rm Pl}$$

$$\hbar = c = G = 1$$

$$\langle 0|T_\mu^\nu|0\rangle_{ren} \sim \delta_\mu^\nu \langle 0|T_k^k|0\rangle_{ren}, \tag{7}$$

$\langle o|T_k^k|0\rangle_{ren}$ is determined by the conformal anomaly for massless conformally invariant fields

Starobinsky A. (Phys. Lett. B, 1980)
Mamayev S. and Mostepanenko V. (Sov. Phys.–JETP, 1980)
Kofman L., Sahni V. and Starobinsky A. (Sov. Phys.–JETP, 1983)
Kofman L. and Sahni V. (Phys. Lett. B, 1983)
Kofman L. and Sahni V. (Phys. Lett. A, 1986)
Spindel P. (Phys. Rev. D, 1988)

Approximate methods of evaluation $\langle 0|T_{\mu\nu}|0\rangle_{ren}$ for massless conformally invariant fields.

Page (Phys. Rev. D, 1982)
Brown, Ottewill (Phys. Rev. D, 1985)
Brown, Ottewill and Page (Phys. Rev. D, 1986)
Frolov, Zel'nikov (Phys. Rev. D, 1987)
Anderson, Hiscock, Samuel (Phys. Rev. D, 1995)
Popov (Phys. Rev. D, 2001)
Groves, Anderson and Carlson (Phys. Rev. D, 2002)

Numerical solutions of a semiclassical theory of gravitation

$$G_{\mu\nu} = 8\pi\langle 0|T_{\mu\nu}|0\rangle_{ren}, \tag{8}$$

which describe wormholes within the framework of approximate methods of calculating $\langle 0|T_{\mu\nu}|0\rangle_{ren}$:

Sushkov (Phys. Lett. A, 1992)
Hochberg, Popov, and Sushkov (Phys. Rev. Lett.1997)

The problem of this approach is the non-availability of limitations of applicability for the approximation of $\langle 0|T_{\mu\nu}|0\rangle_{ren}$

$$\frac{1}{ml} \ll 1, \tag{9}$$

m – mass of the quantized field,
l – characteristic scale of the curvature radius

DeWitt-Schwinger approximation:

$\langle 0|T_{\mu\nu}|0\rangle_{ren}(x)$ is determined by $g_{\mu\nu}(x)$ in a neighborhood of x

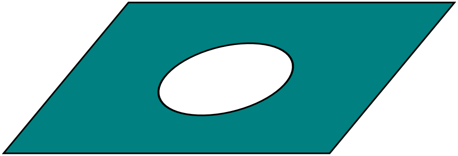
Anderson, Hiscock, Samuel (Phys. Rev. D, 1995)
Matyjasek (Phys. Rev. D, 1999)
Matyjasek (Phys. Rev. D, 2000)
Popov (Phys. Rev. D, 2001)

$$G_{\mu\nu} = 8\pi \left(\langle 0|T_{\mu\nu}|0\rangle_{m=0} + \langle 0|T_{\mu\nu}|0\rangle_{m\gg 1/l} \right)$$

$$\langle 0|T_{\mu\nu}|0\rangle_{m=0} \gg \langle 0|T_{\mu\nu}|0\rangle_{m\gg 1/l} \sim \left(\frac{1}{ml}\right)^2 \langle 0|T_{\mu\nu}|0\rangle_{m=0} \tag{10}$$

Khusnutdinov, Sushkov (2002)

$$ds^2 = -dt^2 + d\rho^2 + r^2(\rho)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$



$$r(\rho) = |\rho| + a,$$

$$R = -\frac{8}{a}\delta(\rho).$$

The total energy of quantum vacuum fluctuations of a massless scalar field

$$E = \int_V -\langle 0|T_t^t|0\rangle^{\text{ren}} \sqrt{g^{(3)}}\, d^3x = -\frac{c^4}{2G}\int_{-\infty}^{\infty} G_t^t\, r^2(\rho)\, d\rho = -\frac{2c^4 a}{G}. \qquad (11)$$

$$a \simeq l_{Pl}, \qquad (12)$$

$l_{Pl} = \sqrt{\hbar G/c^3}$ – Planck length

$$S \sim \int d^4x\sqrt{-g}\left[\frac{R}{8\pi} - \varphi_{,k}\varphi^{,k} - (m^2 + \xi R)\varphi^2\right] \tag{13}$$

$$ds^2 = -f(\rho)dt^2 + d\rho^2 + r^2(\rho)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{14}$$

$-\infty < \rho < \infty$, $f(\rho) > 0$, $r(\rho) > r_{min}$ - radius of throat

Long throat is the region of spacetime where the metric functions $f(\rho), r(\rho)$ are slowly varying:

$$\varepsilon_{\mathrm{WKB}} = \frac{L_\star}{L} \ll 1, \tag{15}$$

$$L_\star = \left[m^2 + \frac{2\xi}{r^2}\right]^{-1/2} \tag{16}$$

$L(\rho)$ – is a characteristic scale of variation of $f(\rho), r(\rho)$:

$$\frac{1}{L(\rho)} = max\left\{\left|\frac{r'}{r}\right|, \left|\frac{f'}{f}\right|\right\} \tag{17}$$

Popov A. (Phys. Rev. D, 2001)

$$ds^2 = -f(\rho)dt^2 + d\rho^2 + r^2(\rho)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{18}$$

$$\begin{aligned}\left\langle T_t^t \right\rangle_{ren} = \langle T_\rho^\rho \rangle_{ren} = \frac{1}{4\pi^2 r^4} \Bigg\{ & \frac{m^2 r^2}{8}\left(\xi - \frac{1}{8}\right) + \frac{79}{7680} - \frac{11}{96}\xi + \frac{3}{8}\xi^2 \\ & + \left[-\frac{m^4 r^4}{8} + \frac{m^2 r^2}{2}\left(\frac{1}{6} - \xi\right) - \frac{1}{60} + \frac{1}{6}\xi - \frac{1}{2}\xi^2\right] \ln\sqrt{\frac{\mu^2}{m_{\mathrm{DS}}^2 r^2}} \\ & + \left[\frac{m^4 r^4}{2} + 2m^2 r^2\left(\xi - \frac{1}{8}\right) + 2\left(\xi - \frac{1}{8}\right)^2\right][I_1(\mu) - I_2(\mu)]\Bigg\} \\ & + O\left(\frac{\varepsilon_{\mathrm{WKB}}^2}{r^4}\right), \end{aligned} \tag{19}$$

$$\left\langle T_\theta^\theta \right\rangle_{ren} = \langle T_\varphi^\varphi \rangle_{ren} = \ldots, \tag{20}$$

$$\mu^2 = m^2 r^2 + 2(\xi - 1/8), \tag{21}$$

$$I_1(\mu) = \int_0^\infty \frac{x \ln|1 - x^2|}{1 + e^{2\pi|\mu|x}} dx, \quad I_2(\mu) = \int_0^\infty \frac{x^3 \ln|1 - x^2|}{1 + e^{2\pi|\mu|x}} dx, \tag{22}$$

Equations of a semiclassical theory of gravity in the long throat of a wormhole

$$G_{\mu\nu} = 8\pi\left[\langle T_{\mu\nu}\rangle_{ren}\left(\, m=0, \xi=1/6\right) + \langle T_{\mu\nu}\rangle_{ren}\left(\, m, \xi\right) + T^{ef}{}_{\mu\nu}\right] \tag{23}$$

$$\left(T^{ef}\right)^{\mu}_{\nu} = \frac{Q^2}{8\pi r^4}\; diag\,(-1,-1,1,1)\,. \tag{24}$$

$$-\frac{1}{8\pi r^2} \simeq \frac{1}{4\pi^2 r^4}\left[0.00310 + \frac{1}{720}\ln\left(m^2_{\mathrm{DS}} r^2\right) + \frac{1}{m^2 r^2}\left(-\frac{\xi^3}{6} + \frac{\xi^2}{12} - \frac{\xi}{60} + \frac{1}{630}\right)\right] - \frac{Q^2}{8\pi r^4}, \tag{25}$$

$$0 \simeq \frac{1}{4\pi^2 r^4}\left[-0.00171 - \frac{1}{720}\ln\left(m^2_{\mathrm{DS}} r^2\right) + \frac{1}{m^2 r^2}\left(\frac{\xi^3}{3} - \frac{\xi^2}{6} + \frac{\xi}{30} - \frac{1}{315}\right)\right] + \frac{Q^2}{8\pi r^4}. \tag{26}$$

$$r^2 \simeq \sqrt{-\frac{1}{\pi m^2}\left(\frac{\xi^3}{3}-\frac{\xi^2}{6}+\frac{\xi}{30}-\frac{1}{315}\right)} \tag{27}$$

$$Q^2 \simeq \frac{2}{\pi}\left[0.00171+\frac{1}{720}\ln\left(\frac{m_{\mathrm{DS}}^2}{m\sqrt{\pi}}\sqrt{-\left(\frac{\xi^3}{3}-\frac{\xi^2}{6}+\frac{\xi}{30}-\frac{1}{315}\right)}\right)\right.$$
$$\left.+\frac{\sqrt{\pi}}{m}\sqrt{-\left(\frac{\xi^3}{3}-\frac{\xi^2}{6}+\frac{\xi}{30}-\frac{1}{315}\right)}\right] \tag{28}$$

A particular solution is

$$\xi = -10^4,\ m^2 = 10^3,\quad r \simeq 101.49\left(\times \mathit{Planck\ length}\right) \tag{29}$$

Let us note that the stress-energy have the needed "exotic" properties to support the throat of a wormhole:

$$p_r = -\varepsilon = \frac{1}{4\pi^2 r^4}\left[0.00310+\frac{1}{720}\ln\left(m_{\mathrm{DS}}^2 r^2\right)\right.$$
$$\left.+\frac{1}{m^2 r^2}\left(-\frac{\xi^3}{6}+\frac{\xi^2}{12}-\frac{\xi}{60}+\frac{1}{630}\right)\right]-\frac{Q^2}{8\pi r^4}<0, \tag{30}$$

p_r is the radial pressure, ε is the energy density.

A. Popov (Class. Quantum Grav., 2005)

In a high-temperature limit, when

$$T \gg 1/l,$$

T – a temperature of thermal state
l – a characteristic scale of the spacetime curvature

$$\langle T_{\mu\nu} \rangle \sim T^4$$

N. Nakazawa and T. Fukuyama (Nucl. Phys. B, 1985)

$$ds^2 = -f(\rho)dt^2 + d\rho^2 + r^2(\rho)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{31}$$

If the field is at temperature T, then the Green's function is periodic in $\tau = -i\,t$ with period $\frac{1}{T}$

$$4\pi^2\left\langle\varphi^2\right\rangle_{ren} = \frac{4\mu^2}{r^2}J\left(a/\mu\right) - \frac{1}{4r^2}\left(2\xi - \frac{1}{4}\right) - \frac{\mu^2}{r^2}I_1(\mu) + \frac{1}{4r^2}\left[m^2r^2 + 2\left(\xi - \frac{1}{6}\right)\right]\ln\left|\frac{m^2r^2 + 2\xi - 1/4}{m^2_{\mathrm{DS}}r^2}\right|, \tag{32}$$

where

$$a = \frac{2\pi Tr}{\sqrt{f}},\ \mu^2 = m^2r^2 + 2\xi - \frac{1}{4}.$$

$$J\left(a/\mu\right) = \int\limits_1^\infty \frac{\sqrt{\eta^2 - 1}}{e^{2\pi\mu\eta/a} - 1}d\eta, \quad I_1(\mu) = \int_0^\infty \frac{\eta\ln\left|1 - \eta^2\right|}{1 + e^{2\pi\mu\eta}}d\eta. \tag{33}$$

- An analytical approximation of the stress-energy tensor of quantized scalar fields in a long throat of a wormhole was obtained.
- The stress-energy tensor $\langle T_{\mu\nu} \rangle_{ren}$ of the vacuum fluctuations in considered space-time does not depend on the choice of quantum state in which the expectation values are taken. Moreover this stress-energy tensor is determined only by the local geometry of space-time.
- The solution in semiclassical theory of gravity which describes the long throat of the wormhole was obtained. Such objects are created by the electrostatic field and the vacuum fluctuations of quantized scalar fields.
- The analytical approximation for $\langle \varphi^2 \rangle_{ren}$ of quantized scalar fields in the background of a long throat was obtained.

Thanks a lot for attention