On stable exponential cosmological solutions with two factor spaces in the Einstein-Gauss-Bonnet model with a \( \Lambda \)-term

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The cosmological model

The action reads as follows:

\[ S = \int_M d^D z \sqrt{|g|} \left\{ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \right\}, \]

where

\[ \mathcal{L}_2[g] = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2 \]

- Gauss-Bonnet term.

The following manifold is considered:

\[ M = \mathbb{R} \times M_1 \times \ldots \times M_n \]

with the cosmological metric

\[ g = -dt \otimes dt + \sum_{i=1}^n B_i e^{2\nu_i t} dy^i \otimes dy^i. \]

\( B_i > 0 \) are arbitrary constants, \( i = 1, \ldots, n \).
The equations of motion for the action leads us to the set of following polynomial equations:

\[
\begin{align*}
G_{ij}v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l &= 0 \\
\left[2G_{ij}v^j - \frac{4}{3}\alpha G_{ijkl} v^j v^k v^l\right] \sum_{k=1}^{n} v^k - \frac{2}{3} G_{sj} v^s v^j + \frac{8}{3} \Lambda &= 0
\end{align*}
\]

where \(\alpha = \alpha_2 / \alpha_1\),

\(G_{ij} = \delta_{ij} - 1\)

\(G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl}\).

For \(\Lambda = 0\) and \(n > 3\) an isotropic solution \(v^1 = \ldots = v^n = H\) exists only if \(\alpha < 0\).

There are no more than 3 different numbers among \(v^1, \ldots, v^n\), when \(\Lambda = 0\).

This is also valid for the case \(\Lambda \neq 0\) when \(\sum_{i=1}^{n} v^i \neq 0\).
Solutions with two Hubble-like parameters

Class of solutions with the following set of Hubble-like parameters:

\[ v = (H, H, H, \underbrace{H, \ldots, H}_m, \underbrace{h, \ldots, h}_l). \]

For an accelerated expansion of a 3-dimensional subspace we put \( H > 0 \).

The \( m \)-dimensional factor space is expanding with the Hubble parameter \( H > 0 \), while the evolution of the \( l \)-dimensional factor space is described by the Hubble-like parameter \( h \).

Restrictions on parameters \( H \) and \( h \):

\[ mH + lh \neq 0, \quad H \neq h \]
Solutions with two Hubble-like parameters

That leads us to the following set of two polynomial equations

\[
\begin{align*}
E &= mH^2 + lh^2 - (mH + lh)^2 + 2\Lambda - \alpha [m(m-1)(m-2)(m-3)H^4
\quad + 4m(m-1)(m-2)lH^3 h + 6m(m-1)l(l-1)H^2 h^2
\quad + 4ml(l-1)(l-2)Hh^3 + l(l-1)(l-2)(l-3)h^4] = 0,
\end{align*}
\]

\[Q = (m-1)(m-2)H^2 + 2(m-1)(l-1)Hh + (l-1)(l-2)h^2 = -\frac{1}{2\alpha}.
\]

Then for \(m > 2\) and \(l > 2\) we get \(H = (-2\alpha \mathcal{P})^{-1/2}\), where

\[\mathcal{P} = \mathcal{P}(x,m,l) = (m-1)(m-2) + 2(m-1)(l-1)x + (l-1)(l-2)x^2, \quad (2.1)\]

\[x = h/H, \quad \alpha \mathcal{P} < 0. \quad (2.2)\]
Solutions with two Hubble-like parameters

So we get the following relation:

\[ \Lambda \alpha = \lambda = \lambda(x, m, l) \equiv \frac{1}{4} (\mathcal{P}(x, m, l))^{-1} \mathcal{M}(x, m, l) \]
\[ + \frac{1}{8} (\mathcal{P}(x, m, l))^{-2} \mathcal{R}(x, m, l), \]  \hspace{1cm} (2.3)

where

\[ \mathcal{M}(x, m, l) \equiv m + lx^2 - (m + lx)^2, \]
\[ \mathcal{R}(x, m, l) \equiv m(m - 1)(m - 2)(m - 3) + 4m(m - 1)(m - 2)lx \]
\[ + 6m(m - 1)l(l - 1)x^2 + 4ml(l - 1)(l - 2)x^3 + l(l - 1)(l - 2)(l - 3)x^4. \]
Solutions with two Hubble-like parameters

The relation (2.1) is valid only if \( \mathcal{P}(x,m,l) \neq 0 \), so:

\[
x \neq x_{\pm} = x_{\pm}(m,l) \equiv \frac{-(m-1)(l-1) \pm \sqrt{\Delta(m,l)}}{(l-1)(l-2)},
\]

\[
\Delta(m,l) \equiv (m-1)(l-1)(m+l-3) = \Delta(l,m).
\]

Here \( x_{\pm}(m,l) \) are roots of the quadratic equation \( \mathcal{P}(x,m,l) = 0 \).

These roots obey the following relations:

\[
x_{+}(m,l)x_{-}(m,l) = \frac{(m-1)(m-2)}{(l-1)(l-2)}, \quad x_{+}(m,l) + x_{-}(m,l) = -2\frac{(m-1)}{l-2},
\]

which lead us to the inequalities \( x_{-}(m,l) < x_{+}(m,l) < 0 \).
Solutions with two Hubble-like parameters

Using (2.2) and (2.3) we get $\Lambda = \alpha^{-1} \lambda(x,m,l)$, where

- $x_-(m,l) < x < x_+(m,l)$ for $\alpha > 0$
- $x < x_-(m,l)$, or $x > x_+(m,l)$ for $\alpha < 0$

For $\alpha < 0$ we have the following limit

$$\lim_{x \to \pm \infty} \lambda(x,m,l) = \lambda_\infty(l) \equiv -\frac{l(l+1)}{8(l-1)(l-2)} < 0.$$ 

Hence

$$\lim_{x \to \pm \infty} \Lambda = \Lambda_\infty \equiv -\frac{l(l+1)}{8\alpha(l-1)(l-2)} > 0, \quad l > 2.$$ 

We note that $\Lambda_\infty$ does not depend upon $m$. 
Solutions with two Hubble-like parameters

For $x = 0$ we get:

$$\Lambda = \Lambda_0 = \alpha^{-1} \lambda(0,m,l) = -\frac{m(m+1)}{8\alpha(m-1)(m-2)} > 0, \quad m > 2.$$  

We see that $\Lambda_0$ does not depend upon $l$.

For $x = 0$ the Hubble-like parameters read

$$H = H_0 = (-2\alpha(m-1)(m-2))^{-1/2}, \quad h = 0$$

and so we get the product of (a part of) $(m + 1)$-dimensional de-Sitter space and $l$-dimensional Euclidean space.
Solutions with two Hubble-like parameters

“Master” equation.

We rewrite eq. (2.3) in the following form

$$2\mathcal{P}(x,m,l)\mathcal{M}(x,m,l) + \mathcal{R}(x,m,l) - 8\lambda(\mathcal{P}(x,m,l))^2 = 0.$$  \hspace{1cm} (2.5)

This equation may be called as a master equation, since the solutions under consideration are governed by it. The master equation is of fourth order in $x$ for $\lambda \neq \lambda_\infty(l)$ or less (of third order for $\lambda = \lambda_\infty(l)$).
Now if we analyze the behaviour of the function $\lambda(x,m,l)$, for fixed $m,l$ and $x \neq x_{\pm}(m,l)$, then we obtain the following extremum points:

$$x_a = 1,$$
$$x_b = x_b(m,l) \equiv -\frac{m-1}{l-2} < 0,$$
$$x_c = x_c(m,l) \equiv -\frac{m-2}{l-1} < 0,$$
$$x_d = x_d(m,l) \equiv -\frac{m}{l} < 0.$$

So for $\lambda_i = \lambda(x_i,m,l), i = a,b,c,d$ we obtain

$$\lambda_a = -\frac{(m+l-1)(m+l)}{8(m+l-3)(m+l-2)} < 0,$$
$$\lambda_b = \frac{lm^2 + (l^2 - 8l + 8)m + l^2 - l}{8(l-2)(m-1)(l+m-3)} > 0,$$
$$\lambda_c = \frac{ml^2 + (m^2 - 8m + 8)l + m^2 - m}{8(m-2)(l-1)(l+m-3)} > 0,$$
$$\lambda_d = \frac{ml(m+l)}{8(lm^2 + ml^2 - 2m^2 - 2l^2 + 2lm)} > 0,$$
We present some examples of the function $\lambda(x) = \Lambda(\alpha)$. At this figures the point $(x_i, \lambda_i)$ is marked by $i$, where $i = a, b, c, d$.

**Figure:** The function $\lambda(x) = \Lambda(\alpha)\alpha$ for $\alpha > 0$, $m = 12$ and $l = 3$.

**Figure:** The function $\lambda(x) = \Lambda(\alpha)\alpha$ for $\alpha > 0$ and $m = l = 4$. 
Bounds on $\Lambda\alpha$ for $\alpha > 0$.
Summarizing all cases we find that for $\alpha > 0$ exact solutions under consideration exist if and only if

$$\Lambda\alpha \leq \begin{cases} 
\lambda_b, & \text{for } m \geq l, \\
\lambda_c, & \text{for } m < l,
\end{cases}$$

(2.6)

Bounds on $\Lambda|\alpha|$ for $\alpha < 0$.
For $\alpha < 0$ exact solutions under consideration exist if and only if

$$\Lambda|\alpha| > |\lambda_a| = \frac{(D - 2)(D - 1)}{8(D - 4)(D - 3)},$$

(2.7)

This relation is valid for all $m > 2, l > 2$ ($D = m + l + 1$), e.g. for $m = 3$. 
Stability analysis

Here we study the stability of exponential solutions with non-static total volume factor, i.e. we put

\[ S_1(v) = \sum_{i=1}^{n} v^i \neq 0. \] (3.1)

Earlier, it was proved that a constant solution \((h^i(t)) = (v^i) (i = 1, \ldots, n; n > 3)\) is stable under perturbations

\[ h^i(t) = v^i + \delta h^i(t), (as \ t \to +\infty) \] (3.2)

in the following case: and it is unstable when

\[ S_1(v) = \sum_{k=1}^{n} v^k > 0 \]
\[ S_1(v) = \sum_{k=1}^{n} v^k < 0. \]


Stability analysis

For our consideration we have \( S_1(v) = mH + lh \).

The perturbations \( \delta h^i \) obey (in the linear approximation) the following set of linear equations:

\[
C_i(v) \delta h^i = 0, \quad (3.3)
\]

\[
L_{ij}(v) \delta h^i = B_{ij}(v) \delta h^j. \quad (3.4)
\]

Here

\[
C_i(v) = 2v_i - 4\alpha G_{ijks} v^i v^j v^k v^s, \quad (3.5)
\]

\[
L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijks} v^k v^s, \quad (3.6)
\]

\[
B_{ij}(v) = -\left( \sum_{k=1}^{n} v^k \right) L_{ij}(v) - L_i(v) + \frac{4}{3} v_j, \quad (3.7)
\]

where \( v_i = G_{ij} v^j \), \( L_i(v) = 2v_i - \frac{4}{3} \alpha G_{ijks} v^j v^k v^s \) and \( i,j,k,s = 1,\ldots,n \).
In our case the set of equations on perturbations (3.3), (3.4) has the following solution

\[ \delta h^i = A^i \exp(-S_1(v)t), \]  
(3.8)

\[ \sum_{i=1}^{n} C_i(v)A^i = 0, \]  
(3.9)

\((A^i \text{ are constants}) \ i = 1, \ldots, n.\)

For the vector \(v\) the matrix \(L\) is a block-diagonal one

\[ (L_{ij}) = \text{diag}(L_{\mu\nu}, L_{\alpha\beta}), \]  
(3.10)

where

\[ L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \]  
(3.11)

\[ L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}) \]  
(3.12)

and

\[ S_{HH} = (m - 2)(m - 3)H^2 + 2(m - 2)lHh + l(l - 1)h^2, \]  
(3.13)

\[ S_{hh} = m(m - 1)H^2 + 2m(l - 2)Hh + (l - 2)(l - 3)h^2. \]  
(3.14)
Stability analysis

The matrix (3.10) is invertible only if $m > 1$, $l > 1$ and

$$S_{HH} \neq -\frac{1}{2\alpha}, \quad S_{hh} \neq -\frac{1}{2\alpha}. \quad (3.15)$$

Inequalities (3.15) are obeyed if $x \neq -\frac{m-2}{l-1} = x_c$ and $x \neq -\frac{m-1}{l-2} = x_b$ for $l > 2$.

In our paper we proved that cosmological solutions under consideration, which obey $x = h/H \neq x_i$, $i = a, b, c, d$, where

$$x_a = 1, \quad x_b = -\frac{m-1}{l-2}, \quad x_c = -\frac{m-2}{l-1}, \quad x_d = -\frac{m}{l},$$

are stable if i) $x > x_d$ and unstable if ii) $x < x_d$. 
**Bounds on $\Lambda\alpha$ for stable solutions with $\alpha > 0$.**

Summarizing all cases presented above we find that for $\alpha > 0$ stable exact solutions under consideration exist if and only if

$$
\Lambda\alpha < \begin{cases} 
\lambda_d, & \text{for } m \geq 2l, \\
\lambda_c, & \text{for } m < 2l,
\end{cases}
$$

(3.16)

where $\lambda_c = \lambda_c(m,l)$ and $\lambda_d = \lambda_d(m,l)$ are defined in (2.6). For $m = 3$ and $l > 2$ we are led to relation $\Lambda\alpha < \lambda_c$.

**Bounds on $\Lambda|\alpha|$ for stable solutions with $\alpha < 0$.**

In the case $\alpha < 0$ we obtain

$$
n_+(\Lambda, \alpha) = \begin{cases} 
1, & \Lambda|\alpha| \geq |\lambda_\infty|, \\
2, & |\lambda_a| < \Lambda|\alpha| < |\lambda_\infty|, \\
0, & \Lambda|\alpha| \leq |\lambda_a|.
\end{cases}
$$

(3.17)

For $\alpha < 0$ stable exact solutions under consideration exist if and only if the relation $(\Lambda|\alpha| > |\lambda_a|)$ is obeyed.
Solutions describing a small enough variation of $G$

Here we analyze the solutions by using the restriction on variation of the effective gravitational constant $G$, which is inversely proportional (in the Jordan frame) to the volume scale factor of the (anisotropic) internal space, i.e.

$$G = \text{const} \exp \left[ -(m - 3)Ht - lht \right]. \quad (4.1)$$

By using (4.1) one can get the following formula for a dimensionless parameter of temporal variation of $G$ ($G$-dot):

$$\delta \equiv \frac{\dot{G}}{GH} = -(m - 3 + lx), \quad x = h/H. \quad (4.2)$$

Here $H > 0$ is the Hubble parameter. Due to observational data, the variation of the gravitational constant is on the level of $10^{-13}$ per year and less.
When the value $\delta$ is fixed we get from (4.2)

$$x = x_0(\delta) = x_0(\delta, m, l) \equiv -\frac{(m - 3 + \delta)}{l}. \quad (4.3)$$

The substitution of $x = x_0(\delta, m, l)$ into quadratic polynomial (2.1) gives us

$$\mathcal{P}(x_0(\delta, m, l), m, l) = \mathcal{P}(x_0(0, m, l), m, l)$$

$$-4 \frac{(l - 1)(m + l - 3)}{l^2} \delta + \frac{(l - 1)(l - 2)}{l^2} \delta^2, \quad (4.4)$$

where

$$\mathcal{P}(x_0(0, m, l), m, l) \equiv \mathcal{P}_0(m, l) = \frac{1}{l^2} (m + l - 3)[(5 - m)l + 2m - 6]. \quad (4.5)$$

We note that equation $\mathcal{P}_0(m, l) = 0$ implies relation

$$l = l_0(m) = \frac{2m-6}{m-5} = 2 + \frac{4}{m-5}, \ m \neq 5.$$

For $m > 9$ we get $2 < l_0(m) < 3$, that means that integer solutions are absent in this interval.
For $3 \leq m \leq 9$ and $m \neq 5$, the only integer values of $l_0(m) > 2$ takes place for $m = 6, 7, 9$ and we get a special set of pairs $(m, l)$:

$$A) \quad (m, l) = (6, 6), (7, 4), (9, 3). \quad (4.6)$$

In the case A) the restriction gives us (see (4.4)) $\delta \neq 0$ and $-4(l + m - 3)\delta + (l - 2)\delta^2 \neq 0$ for $l > 2$ which lead us to two restrictions:

$$\delta \neq 0 \text{ and } \delta \neq 4 \frac{l + m - 3}{l - 2} = 9, 16, 36 \quad (4.7)$$

for $(m, l) = (6, 6), (7, 4), (9, 3)$, respectively.

But the second one may be omitted due to bounds on the value of the dimensionless variation of the effective gravitational constant.
Let us consider the second case

\[ B) \quad m = 5, \quad l > 2. \] (4.8)

In this case the restriction reads

\[ 4(l + 2) - 4(l - 1)(l + 2)\delta + (l - 1)(l - 2)\delta^2 \neq 0, \] (4.9)

\[ l > 2. \] It may be rewritten as

\[ \delta \neq \delta_{\pm}(5,l) \equiv 2\frac{(l + 2)}{(l - 2)} \left( 1 \pm \sqrt{\frac{l^2}{(l - 1)(l + 2)}} \right). \] (4.10)

The first restriction \( \delta \neq \delta_{\pm}(5,l), \ l > 2, \) may be omitted due to the bounds. The second restriction forbids one value of \( \delta \) obeying the bounds for big enough value of \( l \) (e.g., for \( l > 1000 \)).
Now we consider the last case

\[ C \) \( (m,l) \) do not belong to cases A and B. \] (4.11)

In the case C) the restriction reads

\[ (l+m-3)[(5-m)l+2m-6]-4(l-1)(l+m-3)\delta+(l-1)(l-2)\delta^2 \neq 0, \]

\( l > 2 \). It may be rewritten as

\[ \delta \neq \delta \pm (m,l) \equiv 2 \frac{(l+m-3)}{(l-2)} \left(1 \pm \sqrt{\frac{l^2(m-1)}{4(l-1)(l+m-3)}}\right). \] (4.12)

The first restriction \( \delta \neq \delta \pm (m,l) \) \( (l > 2) \) may be omitted due to the bounds since \( \delta \pm (m,l) > 2 \frac{(l+m-3)}{(l-2)} > 2 \) for \( m > 2, l > 2 \).

So, the only second restriction \( \delta \neq \delta -(m,l) \), should be imposed
Now we analyse the stability of these special solutions. The main condition for stability $x_0(\delta) > x_d$ is satisfied since

$$x_0(\delta) - x_d = \frac{3 - \delta}{I} > 0$$  \hspace{1cm} (4.13)

due to our bounds.

Other three conditions are satisfied due to bounds and inequalities: $\delta_a \leq -3$, $\delta_b > 2$ and $\delta_c > 1$. We have shown that all well-defined solutions under consideration, which obey our restrictions and the physical bounds, are stable.

THANK YOU FOR YOUR ATTENTION!
Exact exponential cosmological solutions for $m = l$

Master equation. The master equation reads

For any $m > 2$ the master equation reads

$$Ax^4 + Bx^3 + Cx^2 + Bx + A = 0,$$  \hfill (0.1)

where

$$A = 8\lambda(m - 2)^2(m - 1) + m(m + 1)(m - 2),$$

$$B = 32\lambda(m - 2)(m - 1)^2 + 4m(m - 1)^2,$$

$$C = 16\lambda(m - 1)(3m^2 - 8m + 6) + 2m(m - 1)(3m - 4).$$

It may be solved in radicals, e.g. by using Maple or Mathematica. For $A \neq 0$, or $\lambda \neq \lambda_\infty(m) < 0$ we obtain

$$x = \frac{1}{4A}[-B + \nu_1\sqrt{E - 2B\nu_2\sqrt{d} + \nu_2\sqrt{d}}],$$  \hfill (0.2)

where $\nu_1 = \pm 1$, $\nu_2 = \pm 1$ and

$$d = 8A^2 - 4CA + B^2,$$  \hfill (0.3)

$$E = -8A^2 - 4CA + 2B^2.$$  \hfill (0.4)

The special solution for $m = 3$ was considered recently in ref. $IvKob$. 

Conclusions

In the $D$-dimensional Einstein-Gauss-Bonnet-$\Lambda$ model and two constants $\alpha_1$ and $\alpha_2$ we have found for (fine-tuned) $\Lambda = \Lambda(x, m, l, \alpha)$ with $\alpha = \alpha_2/\alpha_1$ a class of cosmological solutions with exponential time dependence of two scale factors. The solutions are governed by two Hubble-like parameters $H > 0$ and $h$, corresponding to submanifolds of dimensions $m > 2$ and $l > 2$, respectively, with $D = 1 + m + l$. Here $m > 2$, $l > 2$ and parameter $x = h/H$ satisfies the restrictions: $x \neq 1$, $x \neq x_d = -m/l$ and $x \neq x_\pm$.

Any solution describes an exponential expansion of $3d$ subspace with the Hubble parameter $H > 0$ and anisotropic behaviour of $(m - 3 + l)$-dimensional internal space.

The solutions are governed by master equation $\Lambda(x, m, l, \alpha) = \Lambda$, which may be solved in radicals for all values of $\Lambda$ (the case $m = l$ is presented).

Here we have obtained the bounds on $\Lambda$ which guarantee the existence of the exponential (e.g. stable) cosmological solutions under consideration.

We have proved that any of these solutions obeying $x \neq -\frac{m-2}{l-1}$ and $x \neq -\frac{m-1}{l-2}$, is stable (as $t \to +\infty$) if $x > x_d = -m/l$ and unstable if $x < x_d$.

It was also shown that all (well-defined) solutions with small enough variation of the effective gravitational constant $G$ (in the Jordan frame) are stable.