On stable exponential cosmological solutions with two factor spaces in the Einstein-Gauss-Bonnet model with a Λ-term

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The cosmological model

The action reads as follows:

$$S = \int_M d^D z \sqrt{|g|} \{ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \},$$

where

$\mathcal{L}_2[g] = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2$ - Gauss-Bonnet term.

The following manifold is considered:

$M = \mathbb{R} \times M_1 \times \ldots \times M_n$

$M_1, \ldots, M_n$ are one-dimensional manifolds, $\mathbb{R}$ or S^1, $n > 3$.

with the cosmological metric

$g = -dt \otimes dt + \sum_{i=1}^{n} B_i e^{2v^i t} dy^i \otimes dy^i$.

$B_i > 0$ are arbitrary constants, $i = 1, \ldots, n$.

The cosmological model

The equations of motion for the action leads us the to the set of following polynomial equations:

$$\begin{cases} G_{ij} v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l = 0 \\ \left[2 G_{ij} v^j - \frac{4}{3} \alpha G_{ijkl} v^j v^k v^l \right] \sum_{k=1}^{n} v^k - \frac{2}{3} G_{sj} v^s v^j + \frac{8}{3} \Lambda = 0 \end{cases}$$

where $\alpha = \alpha_2/\alpha_1$,

$G_{ij} = \delta_{ij} - 1$

$G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl}$.

For $\Lambda = 0$ and $n > 3$ an isotropic solution $v^1 = \ldots = v^n = H$ exists only if $\alpha < 0$

There are no more than 3 different numbers among $v^1, \ldots, v^n$, when $\Lambda = 0$.

This is also valid for the case $\Lambda \neq 0$ when $\sum_{i=1}^{n} v^i \neq 0$.

Solutions with two Hubble-like parameters

Class of solutions with the following set of Hubble-like parameters:

$$v = (\underbrace{H, H, H}_{\text{``our'' space}}, \underbrace{\overbrace{H, \ldots, H}^{m-3}, \overbrace{h, \ldots, h}^{l}}_{\text{internal space}}).$$

For an accelerated expansion of a 3-dimensional subspace we put $H > 0$.

The m-dimensional factor space is expanding with the Hubble parameter $H > 0$, while the evolution of the l-dimensional factor space is described by the Hubble-like parameter h.

Restrictions on parameters H and h:

$$mH + lh \neq 0,$$
$$H \neq h$$

Solutions with two Hubble-like parameters

That leads us to the following set of two polynomial equations

$$\begin{cases} E = mH^2 + lh^2 - (mH + lh)^2 + 2\Lambda - \alpha[m(m-1)(m-2)(m-3)H^4 \\ \qquad + 4m(m-1)(m-2)lH^3h + 6m(m-1)l(l-1)H^2h^2 \\ \qquad + 4ml(l-1)(l-2)Hh^3 + l(l-1)(l-2)(l-3)h^4] = 0, \\ Q = (m-1)(m-2)H^2 + 2(m-1)(l-1)Hh + (l-1)(l-2)h^2 = -\dfrac{1}{2\alpha}. \end{cases}$$

Then for $m > 2$ and $l > 2$ we get $H = (-2\alpha\mathcal{P})^{-1/2}$, where

$$\mathcal{P} = \mathcal{P}(x,m,l) = (m-1)(m-2) + 2(m-1)(l-1)x + (l-1)(l-2)x^2, \tag{2.1}$$

$$x = h/H, \qquad \alpha\mathcal{P} < 0. \tag{2.2}$$

Solutions with two Hubble-like parameters

So we get the following relation:

$$\Lambda\alpha = \lambda = \lambda(x,m,l) \equiv \frac{1}{4}(\mathcal{P}(x,m,l))^{-1}\mathcal{M}(x,m,l) + \frac{1}{8}(\mathcal{P}(x,m,l))^{-2}\mathcal{R}(x,m,l), \quad (2.3)$$

where

$$\mathcal{M}(x,m,l) \equiv m + lx^2 - (m+lx)^2,$$
$$\mathcal{R}(x,m,l) \equiv m(m-1)(m-2)(m-3) + 4m(m-1)(m-2)lx + 6m(m-1)l(l-1)x^2 + 4ml(l-1)(l-2)x^3 + l(l-1)(l-2)(l-3)x^4.$$

Solutions with two Hubble-like parameters

The relation (2.1) is valid only if $\mathcal{P}(x,m,l) \neq 0$, so:

$$\begin{aligned} x \neq x_{\pm} &= x_{\pm}(m,l) \equiv \frac{-(m-1)(l-1) \pm \sqrt{\Delta(m,l)}}{(l-1)(l-2)}, \\ \Delta(m,l) &\equiv (m-1)(l-1)(m+l-3) = \Delta(l,m). \end{aligned} \tag{2.4}$$

Here $x_{\pm}(m,l)$ are roots of the quadratic equation $\mathcal{P}(x,m,l) = 0$.

These roots obey the following relations:

$$x_{+}(m,l)x_{-}(m,l) = \frac{(m-1)(m-2)}{(l-1)(l-2)}, \qquad x_{+}(m,l) + x_{-}(m,l) = -2\frac{(m-1)}{l-2},$$

which lead us to the inequalities $x_{-}(m,l) < x_{+}(m,l) < 0$.

Solutions with two Hubble-like parameters

Using (2.2) and (2.3) we get $\Lambda = \alpha^{-1}\lambda(x,m,l)$, where

- $x_{-}(m,l) < x < x_{+}(m,l)$ for $\alpha > 0$
- $x < x_{-}(m,l)$, or $x > x_{+}(m,l)$ for $\alpha < 0$

For $\alpha < 0$ we have the following limit

$$\lim_{x \to \pm\infty} \lambda(x,m,l) = \lambda_{\infty}(l) \equiv -\frac{l(l+1)}{8(l-1)(l-2)} < 0.$$

Hence

$$\lim_{x \to \pm\infty} \Lambda = \Lambda_{\infty} \equiv -\frac{l(l+1)}{8\alpha(l-1)(l-2)} > 0, \qquad l > 2.$$

We note that Λ_{∞} does not depend upon m.

Solutions with two Hubble-like parameters

For $x = 0$ we get:

$$\Lambda = \Lambda_0 = \alpha^{-1}\lambda(0,m,l) = -\frac{m(m+1)}{8\alpha(m-1)(m-2)} > 0, \qquad m > 2.$$

We see that Λ_0 does not depend upon l.

For $x = 0$ the Hubble-like parameters read

$$H = H_0 = (-2\alpha(m-1)(m-2))^{-1/2}, \qquad h = 0$$

and so we get the product of (a part of) $(m+1)$-dimensional de-Sitter space and l-dimensional Euclidean space.

Solutions with two Hubble-like parameters

"Master" equation.

We rewrite eq. (2.3) in the following form

$$2\mathcal{P}(x,m,l)\mathcal{M}(x,m,l) + \mathcal{R}(x,m,l) - 8\lambda(\mathcal{P}(x,m,l))^2 = 0. \qquad (2.5)$$

This equation may be called as a *master equation*, since the solutions under consideration are governed by it. The master equation is of fourth order in x for $\lambda \neq \lambda_\infty(l)$ or less (of third order for $\lambda = \lambda_\infty(l)$).

Now if we analyze the behaviour of the function $\lambda(x,m,l)$, for fixed m,l and $x \neq x_{\pm}(m,l)$, then we obtain the following extremum points:

$$\begin{aligned} x_a &= 1, \\ x_b = x_b(m,l) &\equiv -\tfrac{m-1}{l-2} < 0, \\ x_c = x_c(m,l) &\equiv -\tfrac{m-2}{l-1} < 0, \\ x_d = x_d(m,l) &\equiv -\tfrac{m}{l} < 0. \end{aligned}$$

So for $\lambda_i = \lambda(x_i,m,l)$, $i = a,b,c,d$ we obtain

$$\lambda_a = -\frac{(m+l-1)(m+l)}{8(m+l-3)(m+l-2)} < 0,$$

$$\lambda_b = \frac{lm^2 + (l^2 - 8l + 8)m + l^2 - l}{8(l-2)(m-1)(l+m-3)} > 0,$$

$$\lambda_c = \frac{ml^2 + (m^2 - 8m + 8)l + m^2 - m}{8(m-2)(l-1)(l+m-3)} > 0,$$

$$\lambda_d = \frac{ml(m+l)}{8(lm^2 + ml^2 - 2m^2 - 2l^2 + 2lm)} > 0,$$

We present some examples of the function $\lambda(x) = \Lambda\alpha$. At this figures the point (x_i, λ_i) is marked by i, where $i = a,b,c,d$.

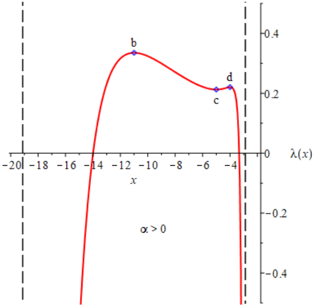


Figure: The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $m = 12$ and $l = 3$.

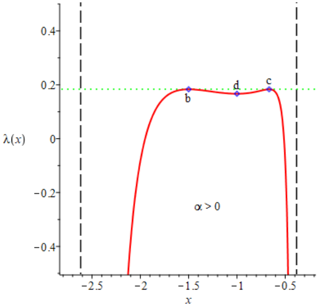


Figure: The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $m = l = 4$.

Solutions with two Hubble-like parameters

Bounds on $\Lambda\alpha$ for $\alpha > 0$.

Summarizing all cases we find that for $\alpha > 0$ exact solutions under consideration exist if and only if

$$\Lambda\alpha \leq \begin{cases} \lambda_b, \text{for } m \geq l, \\ \lambda_c, \text{for } m < l, \end{cases} \tag{2.6}$$

Bounds on $\Lambda|\alpha|$ for $\alpha < 0$.

For $\alpha < 0$ exact solutions under consideration exist if and only if

$$\Lambda|\alpha| > |\lambda_a| = \frac{(D-2)(D-1)}{8(D-4)(D-3)}, \tag{2.7}$$

This relation is valid for all $m > 2$, $l > 2$ ($D = m + l + 1$), e.g. for $m = 3$.

Stability analysis

Here we study the stability of exponential solutions with non-static total volume factor, i.e. we put

$$S_1(v) = \sum_{i=1}^{n} v^i \neq 0. \tag{3.1}$$

Earlier, it was proved that a constant solution $(h^i(t)) = (v^i)$ $(i = 1, \ldots, n;\ n > 3)$ is stable under perturbations

$$h^i(t) = v^i + \delta h^i(t),\ (as\ t \to +\infty) \tag{3.2}$$

in the following case:

$$S_1(v) = \sum_{k=1}^{n} v^k > 0$$

and it is unstable when

$$S_1(v) = \sum_{k=1}^{n} v^k < 0.$$

Stability analysis

For our consideration we have $S_1(v) = mH + lh$.
The perturbations δh^i obey (in the linear approximation) the following set of linear equations:

$$C_i(v)\delta h^i = 0, \tag{3.3}$$

$$L_{ij}(v)\delta \dot{h}^j = B_{ij}(v)\delta h^j. \tag{3.4}$$

Here

$$C_i(v) = 2v_i - 4\alpha G_{ijks} v^j v^k v^s, \tag{3.5}$$

$$L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijks} v^k v^s, \tag{3.6}$$

$$B_{ij}(v) = -(\sum_{k=1}^{n} v^k) L_{ij}(v) - L_i(v) + \frac{4}{3} v_j, \tag{3.7}$$

where $v_i = G_{ij} v^j$, $L_i(v) = 2v_i - \frac{4}{3}\alpha G_{ijks} v^j v^k v^s$ and $i,j,k,s = 1, \dots, n$.

In our case the set of equations on perturbations (3.3), (3.4) has the following solution

$$\delta h^i = A^i \exp(-S_1(v)t), \tag{3.8}$$

$$\sum_{i=1}^{n} C_i(v) A^i = 0, \tag{3.9}$$

(A^i are constants) $i = 1, \dots, n$.
For the vector v the matrix L is a block-diagonal one

$$(L_{ij}) = \mathrm{diag}(L_{\mu\nu}, L_{\alpha\beta}), \tag{3.10}$$

where

$$L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \tag{3.11}$$

$$L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}) \tag{3.12}$$

and

$$S_{HH} = (m-2)(m-3)H^2 + 2(m-2)lHh + l(l-1)h^2, \tag{3.13}$$

$$S_{hh} = m(m-1)H^2 + 2m(l-2)Hh + (l-2)(l-3)h^2. \tag{3.14}$$

Stability analysis

The matrix (3.10) is invertible only if $m > 1$, $l > 1$ and

$$S_{HH} \neq -\frac{1}{2\alpha}, S_{hh} \neq -\frac{1}{2\alpha}. \quad (3.15)$$

Inequalities (3.15) are obeyed if $x \neq -\frac{m-2}{l-1} = x_c$ and $x \neq -\frac{m-1}{l-2} = x_b$ for $l > 2$.

In our paper we proved that cosmological solutions under consideration, which obey $x = h/H \neq x_i$, $i = a,b,c,d$, where

$$x_a = 1, \; x_b = -\frac{m-1}{l-2}, x_c = -\frac{m-2}{l-1}, x_d = -\frac{m}{l},$$

are stable if i) $x > x_d$ and unstable if ii) $x < x_d$.

Bounds on $\Lambda\alpha$ for stable solutions with $\alpha > 0$.

Summarizing all cases presented above we find that for $\alpha > 0$ stable exact solutions under consideration exist if and only if

$$\Lambda\alpha < \begin{cases} \lambda_d, \text{for } m \geq 2l, \\ \lambda_c, \text{for } m < 2l, \end{cases} \tag{3.16}$$

where $\lambda_c = \lambda_c(m,l)$ and $\lambda_d = \lambda_d(m,l)$ are defined in (2.6). For $m = 3$ and $l > 2$ we are led to relation $\Lambda\alpha < \lambda_c$.

Bounds on $\Lambda|\alpha|$ for stable solutions with $\alpha < 0$.

In the case $\alpha < 0$ we obtain

$$n_+(\Lambda, \alpha) = \begin{cases} 1, & \Lambda|\alpha| \geq |\lambda_\infty|, \\ 2, & |\lambda_a| < \Lambda|\alpha| < |\lambda_\infty|, \\ 0, & \Lambda|\alpha| \leq |\lambda_a|. \end{cases} \tag{3.17}$$

For $\alpha < 0$ stable exact solutions under consideration exist if and only if the relation $(\Lambda|\alpha| > |\lambda_a|)$ is obeyed.

Solutions describing a small enough variation of G

Here we analyze the solutions by using the restriction on variation of the effective gravitational constant G, which is inversely proportional (in the Jordan frame) to the volume scale factor of the (anisotropic) internal space, i.e.

$$G = \mathrm{const} \exp\left[-(m-3)Ht - lht\right]. \tag{4.1}$$

By using (4.1) one can get the following formula for a dimensionless parameter of temporal variation of G (G-dot):

$$\delta \equiv \frac{\dot{G}}{GH} = -(m - 3 + lx), \qquad x = h/H. \tag{4.2}$$

Here $H > 0$ is the Hubble parameter. Due to observational data, the variation of the gravitational constant is on the level of 10^{-13} per year and less.

When the value δ is fixed we get from (4.2)

$$x = x_0(\delta) = x_0(\delta, m, l) \equiv -\frac{(m - 3 + \delta)}{l}. \tag{4.3}$$

The substitution of $x = x_0(\delta, m, l)$ into quadratic polynomial (2.1) gives us

$$\mathcal{P}(x_0(\delta, m, l), m, l) = \mathcal{P}(x_0(0, m, l), m, l) \\ -4\frac{(l-1)(m+l-3)}{l^2}\delta + \frac{(l-1)(l-2)}{l^2}\delta^2, \tag{4.4}$$

where

$$\mathcal{P}(x_0(0, m, l), m, l) \equiv \mathcal{P}_0(m, l) = \frac{1}{l^2}(m + l - 3)[(5 - m)l + 2m - 6]. \tag{4.5}$$

We note that equation $\mathcal{P}_0(m, l) = 0$ implies relation
$l = l_0(m) = \frac{2m-6}{m-5} = 2 + \frac{4}{m-5}$, $m \neq 5$.

For $m > 9$ we get $2 < l_0(m) < 3$, that means that integer solutions are absent in this interval.

Solutions describing a small enough variation of G

For $3 \leq m \leq 9$ and $m \neq 5$, the only integer values of $l_0(m) > 2$ takes place for $m = 6,7,9$ and we get a special set of pairs (m,l):

$$A) \quad (m,l) = (6,6), (7,4), (9,3). \tag{4.6}$$

In the case A) the restriction gives us (see (4.4)) $\delta \neq 0$ and $-4(l+m-3)\delta + (l-2)\delta^2 \neq 0$ for $l > 2$ which lead us to two restrictions:

$$\delta \neq 0 \textit{ and } \delta \neq 4\frac{l+m-3}{l-2} = 9, 16{,}36 \tag{4.7}$$

for $(m,l) = (6,6), (7,4), (9,3)$, respectively.
But the second one may be omitted due to bounds on the value of the dimensionless variation of the effective gravitational constant.

Let us consider the second case

$$B) \quad m = 5, \quad l > 2. \tag{4.8}$$

In this case the restriction reads

$$4(l+2) - 4(l-1)(l+2)\delta + (l-1)(l-2)\delta^2 \neq 0, \tag{4.9}$$

$l > 2$. It may be rewritten as

$$\delta \neq \delta_{\pm}(5,l) \equiv 2\frac{(l+2)}{(l-2)}\left(1 \pm \sqrt{\frac{l^2}{(l-1)(l+2)}}\right). \tag{4.10}$$

The first restriction $\delta \neq \delta_{+}(5,l)$, $l > 2$, may be omitted due to the bounds. The second restriction forbids one value of δ obeying the bounds for big anough value of l (e.g., for $l > 1000$).

Now we consider the last case

$$C)\ (m,l)\ \text{do not belong to cases A and B.} \tag{4.11}$$

In the case C) the restriction reads

$$(l+m-3)[(5-m)l+2m-6]-4(l-1)(l+m-3)\delta+(l-1)(l-2)\delta^2 \neq 0,$$

$l > 2$. It may be rewritten as

$$\delta \neq \delta_{\pm}(m,l) \equiv 2\frac{(l+m-3)}{(l-2)}\left(1 \pm \sqrt{\frac{l^2(m-1)}{4(l-1)(l+m-3)}}\right). \tag{4.12}$$

The first restriction $\delta \neq \delta_{+}(m,l)$ ($l > 2$) may be omitted due to the bounds since $\delta_{+}(m,l) > 2\frac{(l+m-3)}{(l-2)} > 2$ for $m > 2$, $l > 2$.
So, the only second restriction $\delta \neq \delta_{-}(m,l)$, should be imposed

Now we analyse the stability of these special solutions.

The main condition for stability $x_0(\delta) > x_d$ is satisfied since

$$x_0(\delta) - x_d = \frac{3 - \delta}{l} > 0 \tag{4.13}$$

due to our bounds.

Other three conditions are satisfied due to bounds and inequalities: $\delta_a \leq -3$, $\delta_b > 2$ and $\delta_c > 1$. We have shown that all well-defined solutions under consideration, which obey our restrictions and the physical bounds, are stable.

THANK YOU FOR YOUR ATTENTION!

Exact exponential cosmological solutions for $m = l$

Master equation. The master equation reads

For any $m > 2$ the master equation reads

$$Ax^4 + Bx^3 + Cx^2 + Bx + A = 0, \tag{0.1}$$

where

$$A = 8\lambda(m-2)^2(m-1) + m(m+1)(m-2),$$

$$B = 32\lambda(m-2)(m-1)^2 + 4m(m-1)^2,$$

$$C = 16\lambda(m-1)(3m^2 - 8m + 6) + 2m(m-1)(3m-4).$$

It may be solved in radicals, e.g. by using Maple or Mathematica. For $A \neq 0$, or $\lambda \neq \lambda_{\infty}(m) < 0$ we obtain

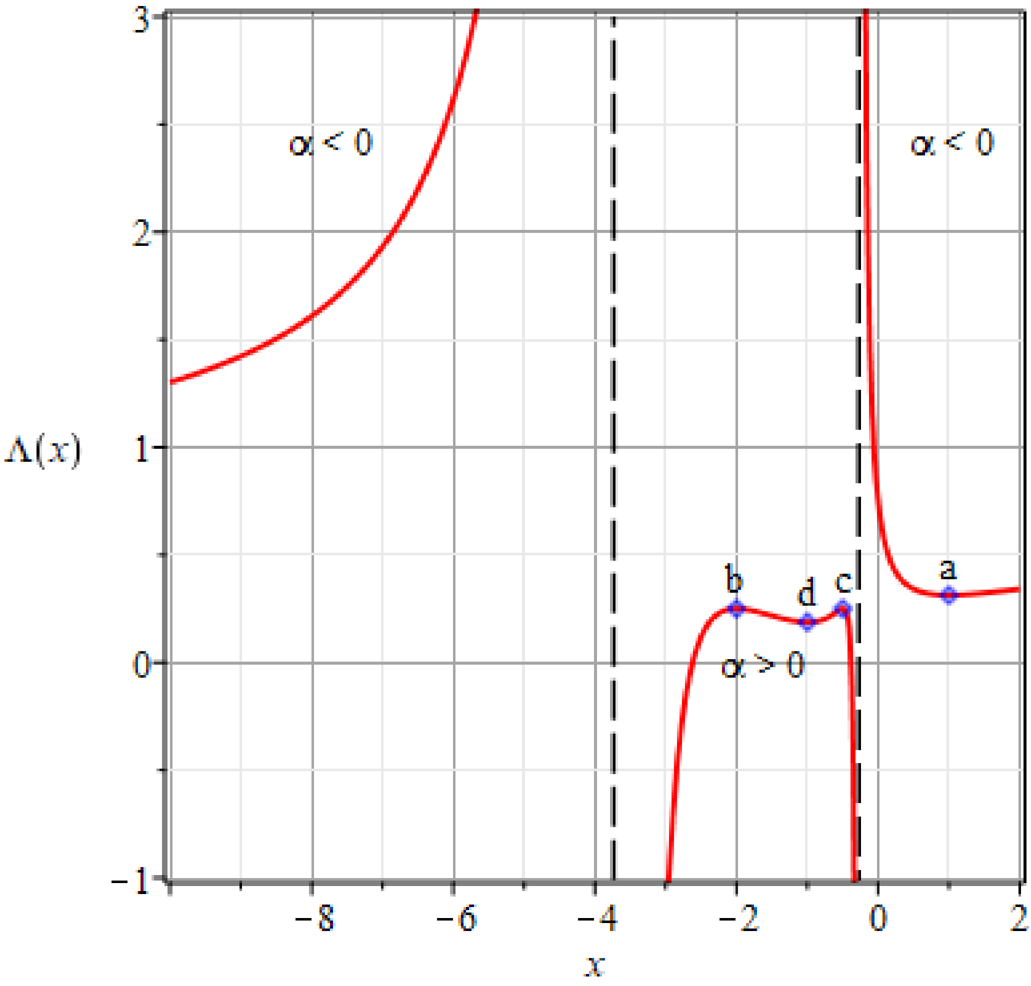
$$x = \frac{1}{4A}[-B + \nu_1\sqrt{E - 2B\nu_2\sqrt{d}} + \nu_2\sqrt{d}], \tag{0.2}$$

where $\nu_1 = \pm 1$, $\nu_2 = \pm 1$ and

$$d = 8A^2 - 4CA + B^2, \tag{0.3}$$

$$E = -8A^2 - 4CA + 2B^2. \tag{0.4}$$

The special solution for $m = 3$ was considered recently in ref. $IvKob$.



3
2
1
0
−1
Λ(x)
α < 0
α < 0
α > 0
a
b
c
d
−8
−6
−4
−2
0
2
x

Conclusions

. In the D-dimensional Einstein-Gauss-Bonnet-Λ model and two constants α_1 and α_2 we have found for (fine-tuned) $\Lambda = \Lambda(x, m, l, \alpha)$ with $\alpha = \alpha_2/\alpha_1$ a class of cosmological solutions with exponential time dependence of two scale factors. The solutions are governed by two Hubble-like parameters $H > 0$ and h, corresponding to submanifolds of dimensions $m > 2$ and $l > 2$, respectively, with $D = 1 + m + l$. Here $m > 2$, $l > 2$ and parameter $x = h/H$ satisfies the restrictions: $x \neq 1$, $x \neq x_d = -m/l$ and $x \neq x_{\pm}$.

. Any solution describes an exponential expansion of $3d$ subspace with the Hubble parameter $H > 0$ and anisotropic behaviour of $(m-3+l)$-dimensional internal space.

. The solutions are governed by master equation $\Lambda(x, m, l, \alpha) = \Lambda$, which may be solved in radicals for all values of Λ (the case $m = l$ is presented).

. Here we have obtained the bounds on Λ which guarantee the existence of the exponential (e.g. stable) cosmological solutions under consideration.

. We have proved that any of these solutions obeying $x \neq -\frac{m-2}{l-1}$ and $x \neq -\frac{m-1}{l-2}$, is stable (as $t \to +\infty$) if $x > x_d = -m/l$ and unstable if $x < x_d$.

. It was also shown that all (well-defined) solutions with small enough varation of the effective gravitational constant G (in the Jordan frame) are stable.