On generalized Melvin's solutions for Lie algebras of rank 3

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Overview

- We consider a multidimensional analogue of Melvin's solution associated with a simple Lie algebra $\mathcal{G}$.
  - The system is a static cylindrically-symmetric gravitational configuration in D dimensions in presence of n Abelian 2-forms and $l \geq n$ scalar fields where n is the rank of $\mathcal{G}$.
  - The solution is governed by n functions $H_s(\rho)$ obeying a set of second-order ODEs with certain boundary conditions. It was conjectured earlier that these functions should be **polynomials** [*Ivashchuk*'2002].
- We obtain the polynomials corresponding to the rank-3 Lie algebras A_3, B_3 and C_3 and reveal some algebraic properties of the solution such as symmetry and duality identities.
- We also find the asymptotic behavior of the solution as well as 2-form flux integrals and corresponding Wilson loop factors.

The model

Let us consider a D-dimensional model governed by the action

$$S = \int d^D x \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} \sum_{s=1}^{n} \mathrm{e}^{2\lambda_s(\varphi)} (F^s)^2 \right\}$$

- $g = g_{MN}(x) dx^M \otimes dx^N$ is a metric; $|g| = |\det(g_{MN})|$
- $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector of l scalar fields
- $(h_{\alpha\beta})$ is a constant symmetric non-degenerate $l \times l$ matrix
- $F^s = dA^s = \frac{1}{2} F^s_{MN} dz^M \wedge dz^N$ is a 2-form field, $s = 1, ..., n$;
  $(F^s)^2 \equiv F^s_{M_1 M_2} F^s_{N_1 N_2} g^{M_1 N_1} g^{M_2 N_2}$
- λ_s is a 1-form on $\mathbb{R}^l$: $\lambda_s(\varphi) = \lambda_{s\alpha} \varphi^\alpha$, $s = 1, ..., n$; $\alpha = 1, \ldots, l$.

Consider a family of exact solutions depending on one variable ρ and defined on the manifold $M = (0, +\infty) \times M_1 \times M_2$.

- M_1 is a one-dimensional manifold (say S^1 or $\mathbb{R}$);
- M_2 is a $(D-2)$-dimensional Ricci-flat manifold.

The solution reads:

$$g = \left(\prod_{s=1}^{n} H_s^{2h_s/(D-2)} \right) \left\{ w d\rho \otimes d\rho + \left(\prod_{s=1}^{n} H_s^{-2h_s} \right) \rho^2 d\phi \otimes d\phi + g^2 \right\}$$

$$F^s = q_s \left(\prod_{s'=1}^{n} H_{s'}^{-A_{ss'}} \right) \rho d\rho \wedge d\phi; \quad \exp(\varphi^\alpha) = \prod_{s=1}^{n} H_s^{h_s \lambda_s^\alpha}$$

where $s = 1, ..., n$; $w = \pm 1$;
$g^1 = d\phi \otimes d\phi$ is a metric on M_1; g^2 is a Ricci-flat metric on M_2.

The general solution

The functions $H_s(z) > 0$, $z \equiv \rho^2$, obey the equations

$$\boxed{\frac{d}{dz}\left(\frac{z}{H_s}\frac{d}{dz}H_s\right) = P_s \prod_{s'=1}^{n} H_{s'}^{-A_{ss'}}}, \qquad H_s(+0) = 1$$

- $P_s = (1/4)K_s q_s^2$, where q_s are integration constants
- $h_s = K_s^{-1}$
- $K_s = B_{ss} > 0$
- $B_{ss'} \equiv 1 + \frac{1}{2-D} + \lambda_{s\alpha}\lambda_{s'\beta}h^{\alpha\beta}, \quad s, s' = 1, ..., n$
- $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}; \quad \lambda_s^{\alpha} = h^{\alpha\beta}\lambda_{s\beta}$
- $A_{ss'} = 2B_{ss'}/B_{s's'}$ is the **Cartan matrix** for a simple Lie algebra $\mathcal{G}$ of rank n.

Special cases

- The solution under consideration is as a special case of the **fluxbrane** (for $w = +1$, $M_1 = S^1$) and S**-brane** ($w = -1$) solutions.
- If $w = +1$ and the Ricci-flat metric g_2 has a pseudo-Euclidean signature, we get a multidimensional generalization of **Melvin's solution** originally describing the gravitational field of a magnetic flux tube [Melvin'1964].
  - Melvin's solution (without scalar field) corresponds to $D = 4$, $n = 1$, $M_1 = S^1$ $(0 < \phi < 2\pi)$, $M_2 = \mathbb{R}^2$, $g_2 = -dt \otimes dt + d\xi \otimes d\xi$ and $\mathcal{G} = A_1$.
- For $w = -1$ and g_2 of Euclidean signature we obtain a **cosmological solution** with a horizon (as $\rho = +0$) if $M_1 = \mathbb{R}$ $(-\infty < \phi < +\infty)$

The Polynomial Conjecture

According to a conjecture [Ivashchuk'2002], the solutions governed by the Cartan matrix $(A_{ss'})$ are **polynomials** (the so-called fluxbrane polynomials):

$$\boxed{H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k}$$

- $n_s = 2 \sum_{s'=1}^{n} A^{ss'}$ – components of a twice dual Weyl vector in the basis of simple co-roots
- $(A^{ss'}) \equiv (A_{ss'})^{-1}$ – the inverse Cartan matrix.
- $P_s^{(k)}$ are constants ($P_s^{(1)} \equiv P_s$). Here $P_s^{(n_s)} \neq 0$. Below we will use the rescaled variables $p_s = P_s / n_s$.

In previous works this conjecture was verified for some cases of the classic Lie algebra series as well as for exceptional algebras G_2, E_6. Here we analyze the solution in case of rank-3 Lie algebras A_3, B_3 and C_3 in more detail.

Case of the rank-3 Lie algebras

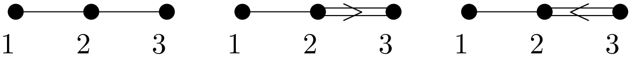
We deal with the solution for $n = l = 3$, $w = +1$ and $M_1 = S^1$. We put here $h_{\alpha\beta} = \delta_{\alpha\beta}$ and denote $(\lambda_{sa}) = (\lambda_s^a) = \vec{\lambda}_s$, $s = 1, \dots, 3$.

For $\mathcal{G} = A_3, B_3, C_3$ respectively we have:

- The Cartan matrices

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

- The Dynkin diagrams:



- The degrees of the fluxbrane polynomials $H_s(z)$:

$(n_1, n_2, n_3) = (3, 4, 3), \ (6, 10, 6), \ (5, 8, 9)$

The polynomials $H_s(z)$ for the 3-rank Lie algebras

We can write the explicit form of the fluxbrane polynomials in case of Lie algebras A_3, B_3, C_3.

The case $A_3 \cong sl(4)$

$$H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1 p_2 p_3 z^3$$
$$H_2 = 1 + 4p_2 z + (3p_1 p_2 + 3p_2 p_3) z^2 + 4p_1 p_2 p_3 z^3 + p_1 p_2^2 p_3 z^4$$
$$H_3 = 1 + 3p_3 z + 3p_2 p_3 z^2 + p_1 p_2 p_3 z^3$$

The polynomials $H_s(z)$ for the 3-rank Lie algebras

The case $B_3 \cong so(7)$

$$
\begin{aligned}
H_1 &= 1 + 6p_1 z + 15p_1p_2 z^2 + 20p_1p_2p_3 z^3 + 15p_1p_2p_3^2 z^4 + 6p_1p_2^2p_3^2 z^5 \\
&\quad + p_1^2p_2^2p_3^2 z^6 \\
H_2 &= 1 + 10p_2 z + (15p_1p_2 + 30p_2p_3)\, z^2 + \left(80p_1p_2p_3 + 40p_2p_3^2\right) z^3 \\
&\quad + \left(50p_1p_2^2p_3 + 135p_1p_2p_3^2 + 25p_2^2p_3^2\right) z^4 + 252p_1p_2^2p_3^2 z^5 \\
&\quad + \left(25p_1^2p_2^2p_3^2 + 135p_1p_2^3p_3^2 + 50p_1p_2^2p_3^3\right) z^6 + \left(40p_1^2p_2^3p_3^2 + 80p_1p_2^3p_3^3\right) z^7 \\
&\quad + \left(30p_1^2p_2^3p_3^3 + 15p_1p_2^3p_3^4\right) z^8 + 10p_1^2p_2^3p_3^4 z^9 + p_1^2p_2^4p_3^4 z^{10} \\
H_3 &= 1 + 6p_3 z + 15p_2p_3 z^2 + \left(10p_1p_2p_3 + 10p_2p_3^2\right) z^3 + 15p_1p_2p_3^2 z^4 \\
&\quad + 6p_1p_2^2p_3^2 z^5 + p_1p_2^2p_3^3 z^6
\end{aligned}
$$

The polynomials $H_s(z)$ for the 3-rank Lie algebras

The case $C_3 \cong sp(3)$

$$H_1 = 1 + 5p_1z + 10p_1p_2z^2 + 10p_1p_2p_3z^3 + 5p_1p_2^2p_3z^4 + p_1^2p_2^2p_3z^5$$

$$H_2 = 1 + 8p_2z + \left(10p_1p_2 + 18p_2p_3\right)z^2 + \left(40p_1p_2p_3 + 16p_2^2p_3\right)z^3 + 70p_1p_2^2p_3z^4 + \left(16p_1^2p_2^2p_3 + 40p_1p_2^3p_3\right)z^5 + \left(18p_1^2p_2^3p_3 + 10p_1p_2^3p_3^2\right)z^6 + 8p_1^2p_2^3p_3^2z^7 + p_1^2p_2^4p_3^2z^8$$

$$H_3 = 1 + 9p_3z + 36p_2p_3z^2 + \left(20p_1p_2p_3 + 64p_2^2p_3\right)z^3 + \left(90p_1p_2^2p_3 + 36p_2^2p_3^2\right)z^4 + \left(36p_1^2p_2^2p_3 + 90p_1p_2^2p_3^2\right)z^5 + \left(64p_1^2p_2^2p_3^2 + 20p_1p_2^3p_3^2\right)z^6 + 36p_1^2p_2^3p_3^2z^7 + 9p_1^2p_2^4p_3^2z^8 + p_1^2p_2^4p_3^3z^9.$$

- The **asymptotical behaviour** of the polynomials as $z \to \infty$:

$$H_s = H_s(z, (p_i)) \sim \left(\prod_{l=1}^{3} (p_l)^{\nu^{sl}} \right) z^{n_s} \equiv H_s^{\mathrm{as}}(z, (p_i)); \quad \sum_{s=1}^{3} \nu^{sl} = n_l.$$

- (ν^{sl}) are components of the integer valued matrix $\nu(\mathcal{G})$:

$$\nu(A_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ \nu(B_3) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 1 & 2 & 3 \end{pmatrix}, \ \nu(C_3) = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}$$

$$\boxed{\nu(\mathcal{G}) = A^{-1}(\mathcal{G})(I + P(\mathcal{G}))}, \quad P(\mathcal{G}) = (P^i_j) = \begin{cases} \delta^i_{\sigma(j)}, & \mathcal{G} = A_3 \\ \delta^i_j, & \mathcal{G} = B_3, C_3 \end{cases}$$

- Here we introduced a **permutation** $\sigma : (1, 2, 3) \mapsto (3, 2, 1)$ of the vertices of the Dynkin diagram. This is a generator of the symmetry group $G = \{\sigma, \mathrm{id}\} \cong \mathbb{Z}_2$ of the diagram for A_3 case.

Symmetry and duality

Define the <u>dual ordered set</u> of variables $\hat{p}_i$, $i = 1, 2, 3$:

- $\hat{p}_i = p_{\sigma(i)}$ for the A_3 case
- $\hat{p}_i = p_i$ for B_3 and C_3 cases

The following properties hold true:

Proposition 1 (the symmetry relations)

For all p_i and z in case of $\mathcal{G} = A_3$

$$H_{\sigma(s)}(z, (p_i)) = H_s(z, (\hat{p}_i)), \quad s = 1, 2, 3$$

Proposition 2 (the duality relations)

For all $p_i \neq 0$ and $z \neq 0$ in case of $\mathcal{G} = A_3, B_3, C_3$

$$H_s(z, (p_i)) = H_s^{\text{as}}(z, (p_i)) H_s(z^{-1}, (\hat{p}_i^{-1})), \quad s = 1, 2, 3.$$

The solution in case of rank-3 Lie algebras

The manifold: $M = (0, +\infty) \times M_1 \times M_2$.

- $M_1 = S^1$ $(0 < \phi < 2\pi)$;
- M_2 is a $(D-2)$-dimensional Ricci-flat manifold.

The solution:

$$g = \Big(\prod_{s=1}^{3} H_s^{2h_s/(D-2)}\Big)\Big\{d\rho \otimes d\rho + \Big(\prod_{s=1}^{3} H_s^{-2h_s}\Big)\rho^2 d\phi \otimes d\phi + g^2\Big\}$$

$$F^s = \mathcal{B}^s \rho d\rho \wedge d\phi; \quad \exp(\varphi^a) = \prod_{s=1}^{3} H_s^{h_s \lambda_s^a}; \qquad a, s = 1, \dots, 3$$

$$\mathcal{B}^s = q_s \left(\prod_{l=1}^{3} H_l^{-A_{sl}}\right); \quad K_s = \frac{D-3}{D-2} + \vec{\lambda}_s^2, \quad \vec{\lambda}_s \vec{\lambda}_l = \frac{1}{2} K_l A_{sl} - \frac{D-3}{D-2},$$

$$h_s = K_s^{-1} = \underbrace{(h,h,h)}_{\text{for } A_3}, \ \underbrace{(h,h,2h)}_{\text{for } B_3}, \ \underbrace{(h,h,h/2)}_{\text{for } C_3}$$

The fluxes

Consider 2-dimensional manifold $M_R = (0, R) \times S^1$, $R > 0$.

- The **flux integral** over M_R:

$$\Phi^s(R) = \int_{M_R} F^s = 2\pi \int_0^R d\rho \rho \mathcal{B}^s = 4\pi q_s P_s^{-1} \frac{R^2 H_s'(R^2)}{H_s(R^2)}$$

- The **total flux** is convergent: $\Phi^s = \Phi^s(+\infty) = \boxed{4\pi n_s q_s^{-1} h_s}$
  - Any (total) flux Φ^s depends upon one integration constant $q_s \neq 0$ (while the integrand form F^s depends upon all constants q_1, q_2, q_3)
  - In case of $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$ the parameter q_s is coinciding up to a sign with the value of the x-component of the magnetic field on the axis of symmetry

- There exist (globally defined on $\mathbb{R}^2$) forms A^s: $dA^s = F^s$.
  The **Wilson loop factors** over circle C_R can be calculated:

$$W^s(C_R) = \exp(i \int_{C_R} A^s) = \exp(i\Phi^s(R)); \quad \lim_{R \to +\infty} W^s(C_R) = \exp(i\Phi^s)$$

Asymptotic behaviour

The asymptotic relations for the solution for $\rho \to +\infty$ read

$$g_{\rm as} = \left(\prod_{l=1}^{3} p_l^{a_l}\right)^{\frac{2}{D-2}} \rho^{2A} \left\{ d\rho \otimes d\rho + \left(\prod_{s=1}^{3} p_l^{a_l}\right)^{-2} \rho^{2-2A(D-2)} d\phi \otimes d\phi + g^2 \right\}$$

$$\varphi^a_{\rm as} = \sum_{s=1}^{3} h_s \lambda^a_s \left(\sum_{l=1}^{3} \nu^{sl} \ln p_l + 2 n_s \ln \rho \right)$$

$$F^s_{\rm as} = q_s p_s^{-1} p_{\theta(s)}^{-1} \rho^{-3} d\rho \wedge d\phi, \qquad a, s = 1, \dots, 3$$

where we put $\theta = \sigma$ for $\mathcal{G} = A_3$, and $\theta = {\rm id}$ for $\mathcal{G} = B_3, C_3$;

$$a_l = \sum_{s=1}^{3} h_s \nu^{sl}, \qquad A = 2(D-2)^{-1} \sum_{s=1}^{3} n_s h_s.$$

Conclusions

- We have considered a generalization of the Melvin's solution associated to simple finite-dimensional Lie algebras of rank 3: $\mathcal{G} = A_3, B_3, C_3$.
- Any solution is governed by a set of 3 fluxbrane polynomials $H_s(z)$, $s = 1, 2, 3$.
- The symmetry and duality identities for polynomials are proved, which may be used in deriving $1/\rho$-expansion for solutions at large distances.
- Asymptotic behaviour of the solutions is also found.
- 2D flux integrals and corresponding Wilson loop factors are calculated, their convergence is demonstrated.
- Another possible application of the solutions considered is to study cosmological analogues of such solutions with phantom scalar fields.

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Thank you for your attention!