Nonstationary self-gravitating configurations of scalar and electromagnetic fields

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Action and stress-energy tensor

The action of the gravitating system of a nonlinear real scalar field and the electromagnetic field, which assumed both to be minimally coupled to gravity, is

$$\Sigma = \int \left(-\frac{1}{2} S + \mathcal{L}_{\phi} - \frac{1}{2} F_{ij} F^{ij} \right) \sqrt{|g|} d^4 x,$$

$$\mathcal{L}_{\phi} = \varepsilon \langle d\phi, d\phi \rangle - 2V(\phi),$$

where $F = F_{ij} dx^i \wedge dx^j$- the electromagnetic field tensor, ε= $\pm$1 .

The components of the stress-energy tensor are determined by the formulas

$$T = T_{(\phi)} + T_{(em)} \ ,$$

$$T_{(\phi)ij} = 2\varepsilon \partial_i \phi \partial_j \phi - (\varepsilon g^{km} \partial_k \phi \partial_m \phi - 2V) g_{ij} \ ,$$

$$T_{(em)ij} = -2 g_{ik} F_{jl} F^{kl} + \frac{1}{2} g_{ij} F_{kl} F^{kl} \ .$$

Einstein, Klein-Gordon and Maxwell equations

$$R_{ij} - \frac{1}{2} S g_{ij} = T_{ij} \ ,$$

$$\frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g} g^{ij} \partial_j \phi \right) + \varepsilon V'_\phi = 0, \qquad g = -\det\left(g_{ij}\right),$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} F^{ij} \right) = 0. \tag{1}$$

Since the space-time is spherically symmetric, the electromagnetic field tensor can be written in the following form

$$F = F_{tr} dt \wedge dr, \qquad F_{tr} = F_{tr}(t, r).$$

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Metric and orthonormal basis

We write the spherically symmetric spacetime metric in the form

$$g = A^2 dt \otimes dt - B^2 dr \otimes dr - C^2 (d\theta \otimes d\theta + sin^2\theta d\varphi \otimes d\varphi).$$

It is easy to obtain directly from the formula (1) the expression

$$F = \frac{ABq}{C^2} dt \wedge dr,$$

where q is electric charge.

In orthonormal basis:

$$Adt = e^0, \qquad Bdr = e^1, \qquad Cd\theta = e^2, \qquad C \sin\theta \, d\varphi = e^3,$$

$$F = \frac{q}{C^2} e^0 \wedge e^1.$$

Einstein equations and characteristic function

$$-2\frac{C_{(1)(1)}}{C}+2\frac{B_{(0)}C_{(0)}}{BC}-\frac{C_{(1)}^2-C_{(0)}^2-1}{C^2}=\varepsilon\left(\phi_{(1)}^2+\phi_{(0)}^2\right)+2V+\frac{q^2}{C^4},$$

$$-2\frac{C_{(0)(0)}}{C}+2\frac{A_{(1)}C_{(1)}}{AC}+\frac{C_{(1)}^2-C_{(0)}^2-1}{C^2}=\varepsilon\left(\phi_{(1)}^2+\phi_{(0)}^2\right)-2V-\frac{q^2}{C^4},$$

$$\frac{A_{(1)(1)}}{A}-\frac{B_{(0)(0)}}{B}+\frac{C_{(1)(1)}}{C}-\frac{C_{(0)(0)}}{C}+\frac{A_{(1)}C_{(1)}}{AC}-\frac{B_{(0)}C_{(0)}}{BC}=\varepsilon\left(\phi_{(0)}^2-\phi_{(1)}^2\right)-2V+\frac{q^2}{C^4},$$

$$-2\frac{C_{(0)(1)}}{C}+2\frac{B_{(0)}C_{(1)}}{BC}\equiv-2\frac{C_{(1)(0)}}{C}+2\frac{A_{(1)}C_{(0)}}{AC}=2\varepsilon\phi_{(0)}\phi_{(1)},$$

$$\phi_{(0)(0)}-\phi_{(1)(1)}+\phi_{(0)}\frac{(BC^2)_{(0)}}{BC^2}-\phi_{(1)}\frac{(AC^2)_{(1)}}{AC^2}+\varepsilon V'_{\phi}=0.$$

Consider the function $f=C_{(1)}^2-C_{(0)}^2=-\langle dC,dC\rangle$.

The solutions of the equation *f(C) = 0* define hypersurfaces on which the 1-form dC becomes null. In particular, it is true on event horizons and hence the function *f(C)* will be referred to as the characteristic function.

Approach to constructing nonstationary configurations

For nonstationary scalar field configuration

$$\phi \neq \phi(C),$$

we separate one invariant equation, written in terms of the characteristic function and scalar field potential

$$d[C(f-1)] = C^2\left(\varepsilon\left(\phi_{(1)}^2 - \phi_{(0)}^2\right) - 2V - \frac{q^2}{C^4}\right)dC + 2C^2\varepsilon\left(C_{(0)}\phi_{(0)} - C_{(1)}\phi_{(1)}\right)d\phi,$$

where

$$C_{(0)}\phi_{(0)} - C_{(1)}\phi_{(1)} = \langle d\phi, dC\rangle, \qquad \phi_{(1)}^2 - \phi_{(0)}^2 = -\langle d\phi, d\phi\rangle.$$

The latter, in turn, gives

$$\langle d\phi, dC\rangle = \frac{\varepsilon}{2C}\partial_\phi f, \quad \langle d\phi, d\phi\rangle = -\varepsilon\left(2V + \frac{1}{C}\partial_C f + \frac{f-1}{C^2} + \frac{q^2}{C^4}\right).$$

Approach to constructing nonstationary configurations

A significant part of the proposed method is the use of coordinates

$$(\phi, C, \theta, \varphi),$$

which is possible, at least locally, for any nonstationary configuration.

We have

$$g^{\phi\phi} = \langle d\phi, d\phi \rangle, \qquad g^{C\phi} = \langle dC, d\phi \rangle, \qquad g^{CC} = \langle dC, dC \rangle.$$

Finding the inverse matrix, we obtain the components of the covariant metric tensor and write the metric in coordinates $(\phi, C, \theta, \varphi)$:

$$ds^2 = -4\frac{\varepsilon C^4 f d\phi^2 + C^3 f'_\phi dC d\phi + (C^3 f'_C + C^2(f-1) + 2C^4 V + q^2) dC^2}{4fC^2(2C^2 V + Cf'_C + f - 1) - \varepsilon\left(Cf'_\phi\right)^2 + 4fq^2} -$$

$$-C^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Approach to constructing nonstationary configurations

The characteristic function and the scalar field potential turn out to be connected by one single Klein-Gordon equation, which takes the form

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi\right)+\varepsilon V'_\phi=0 \quad \Leftrightarrow$$

$$\varepsilon=1: \quad 8V^2C^8+\left(6C^7f'_C+8C^6f-2C^6f''_\phi+8C^4q^2-8C^6\right)V+C^6V'_\phi f'_\phi+$$

$$+(-fC^4-C^2q^2-f'_CC^5+C^4)f''_\phi-f''_CC^6f+C^5f''_{\phi C}f'_\phi+(f'_C)^2C^6+(-3C^5+$$

$$+3C^3q^2+3C^5f)f'_C+4f^2C^4+(8C^2q^2-6C^4)f+2C^4+2q^4-4C^2q^2=0.$$

Coordinate system (t, C, θ, φ) :

$$ds^2=-\frac{4C^4f\,dt^2}{\left(4fC^2(2C^2V+Cf'_C+f-1)-\left(Cf'_\phi\right)^2+4fq^2\right)\left(t'_\phi\right)^2}-\frac{dC^2}{f}-$$

$$-C^2(d\theta^2+\sin^2\theta\, d\varphi^2), \qquad \frac{t'_C}{t'_\phi}=\frac{f'_\phi}{2Cf}$$

Special case for metric function

We assume

$$\varepsilon = -1,$$

$$f(\phi, C) = 1 + C^2 h(\phi).$$

Choosing the characteristic function in this form, we obtain an explicit solution for the scalar field potential

$$V(\phi) = -\frac{3h}{2} - \frac{(h'_\phi)^2}{8h} - \frac{e^F (h'_\phi)^2}{8h^2 \int \frac{e^F h'_\phi}{h^2} \, d\phi}, \quad F(\phi) = -4\int \frac{h}{h'_\phi} \, d\phi$$

This choice of the characteristic function is not accidental. It is due to the possibility of obtaining exact solutions with a nontrivial topology of space-time.

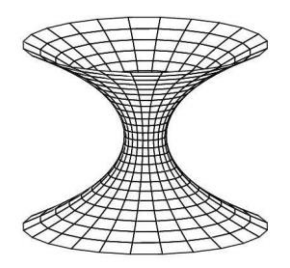
An exact nonstationary solution with a nontrivial topology of space-time

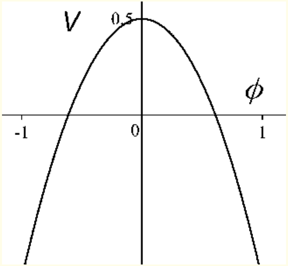
Using the proposed method, we construct a model of nonstationary wormhole.

$$h(\phi) = \phi^2 - 1$$

$$V(\phi) = 1 - \frac{3}{2}\phi^2 + \frac{1 + \phi^2 - e^{\phi^2}}{2\left(1 + e^{\phi^2}(\phi^2 - 1)\right)}.$$

The integration constant was chosen so that the scalar field potential on the wormhole throat ($\phi = 0$) is a regular function.





An exact nonstationary solution with a nontrivial topology of space-time

The exact form of the metric in the coordinates $(\phi, C, \theta, \varphi)$

$$ds^2 = \frac{1}{\Delta}\left(\left(C^2\left(1-\phi^2\right)-1\right)d\phi^2 + 2\phi C d\phi\, dC + \frac{\phi^2\left(1-e^{\phi^2}\right)}{1+e^{\phi^2}(\phi^2-1)}dC^2\right) - C^2 d\Omega^2,$$

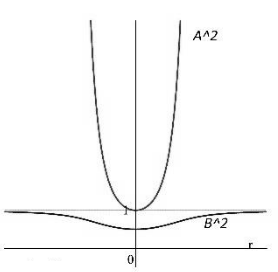
$$\Delta = \frac{\phi^2\left(e^{\phi^2} - \phi^2 C^2 - 1\right)}{1+e^{\phi^2}(\phi^2-1)}.$$

Accordingly, the solution is defined in area

$$e^{\phi^2} - \phi^2 C^2 - 1 < 0.$$

An exact nonstationary solution with a nontrivial topology of space-time

Next, we move on to the ordinary coordinates (t, r, θ, φ).

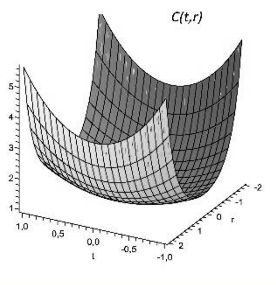


$$\phi = r, \quad C = \operatorname{ch} t \sqrt{\frac{e^{r^2} - 1}{r^2}} = \operatorname{ch} t \left(1 + \frac{r^2}{4} + O(r^4)\right), \quad r \to 0.$$

$$A^2 = \frac{e^{r^2} - 1}{r^2} = 1 + \frac{r^2}{2} + O(r^4), \; r \to 0;$$

$$B^2 = \frac{e^{r^2}\left(r^2 - 1\right) + 1}{r^2\left(e^{r^2} - 1\right)} = \frac{1}{2} + \frac{r^2}{12} + O(r^4), \; r \to 0.$$

Metric functions are regular, positive and even with respect to r.The coordinate r is space-like everywhere. This makes it possible to interpret the solution as a wormhole, the size of which $C(t,0) = \operatorname{ch} t$ varies over time, taking the smallest value $C = 1$ at $t = 0$.



Thank you for attention!