Spherically symmetric double layers and thin shells in Weyl + Einstein gravity

Victor Berezin

The work is done in collaboration with
V. Dokuchaev, Yu. Eroshenko & A. Smirnov

Institute for Nuclear Research of the Russian Academy of Sciences, Moscow

I.1 General scheme

Preliminaries

- Thin shell $=$ 3-dim hypersurface,
  where the energy-momentum tensor is concentrated

$$T_{\mu\nu}\Big|_{\Sigma} = S_{\mu\nu}\delta(\Sigma) \quad \Rightarrow \quad \text{Dirac's } \delta\text{-function}$$

  Here we will consider only timelike shells

- Spherical symmetry $\to$ simplest generalization of a point mass
  Main advantage — backreaction $\to$ self-consistency
  Metric $(\mu, \nu = 0, 1, 2, 3)$, $(i, k = 0, 1)$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ik}dx^i dx^k - r^2(x)(d\theta^2 + \sin^2\theta d\phi^2)$$

- Conformal transformation

$$ds^2 = \Omega^2 d\hat{s}^2 = r^2\left(\tilde{\gamma}_{ik}dx^i dx^k - (d\theta^2 + \sin^2\theta d\phi^2)\right)$$

- $(2+2)$ — decomposition

I.2 Some formulae

- Riemann curvature tensor: $R^{\mu}_{\nu\lambda\sigma}$
- Ricci tensor: $R^{\mu}_{\nu} = R^{\lambda}_{\mu\lambda\nu}$
- Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
- Weyl tensor (completely traceless):

$$
\begin{aligned}
C_{\mu\nu\lambda\sigma} \quad = \quad & R_{\mu\nu\lambda\sigma} + \frac{1}{2}(R_{\mu\sigma}g_{\nu\lambda} + R_{\nu\lambda}g_{\mu\sigma} - R_{\mu\lambda}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\lambda}) \\
& + \frac{1}{6} R (g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\sigma}g_{\nu\lambda})
\end{aligned}
$$

- Bach tensor: $B_{\mu\nu} = C_{\mu\lambda\nu\sigma}{}^{;\sigma;\lambda} + \frac{1}{2} R^{\lambda\sigma} C_{\mu\lambda\nu\sigma}$

$$
B^{\lambda}_{\lambda} = 0, \quad B_{\mu\nu} = B_{\nu\mu}, \quad B^{\lambda}_{\mu;\lambda} = 0
$$

I.3 Conformal transformation and $(2+2)$ – decomposition

- Einstein tensor

$$G_{\mu\nu} = \hat{G}_{\mu\nu} - \frac{2r_{\mu;\nu}}{r} + \frac{2r^{\lambda}{}_{;\lambda}r}{r}\hat{g}_{\mu\nu} + \frac{4r_{\mu}r_{\nu}}{r^2} - \frac{r^{\lambda}r_{\lambda}}{r^2}\hat{g}_{\mu\nu}$$

$$g_{\mu\nu} = r^2\hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{r^2}\hat{g}^{\mu\nu}, \quad r_{\mu} = r_{,\mu}, \quad r^{\lambda} = \hat{g}^{\lambda\sigma}r_{,\sigma}$$

";" — covariant derivative with respect to $\hat{g}_{\mu\nu}$

$$G_{ik} = -\frac{2r_{i|k}}{r} + \frac{4r_i r_k}{r^2} + \left(1 + \frac{2r^{p}{}_{|p}}{r} - \frac{r^p r_p}{r^2}\right)\tilde{\gamma}_{ik},$$

$$G = G^{\lambda}_{\lambda} = -R = -\frac{1}{r^2}\left(-\hat{R} + \frac{6r^{l}{}_{|l}}{r}\right), \quad \hat{R} = \tilde{R} - 2$$

$\tilde{R}$ — scalar curvature of 2-dim space-time with the metric $\tilde{\gamma}_{ik}$
"|" — covariant derivative with respect to $\tilde{\gamma}_{ik}$

I.4 Conformal transformation and $(2+2)$ – decomposition

- Bach tensor

$$B_{\mu\nu} = \frac{1}{r^2}\hat{B}_{ik}$$

$$\hat{B}_{ik} = -\frac{1}{6}\left(\tilde{R}^{|p}_{|p}\tilde{\gamma}_{ik} - \tilde{R}_{|ik} + \frac{\tilde{R}^2 - 4}{4}\tilde{\gamma}_{ik}\right)$$

$$\hat{B}^{\mu}_{\mu} = 0 \qquad \Rightarrow \qquad \hat{B}^2_2 = \hat{B}^3_3 = -\frac{1}{2}\hat{B}^l_l$$

I.5 Gauss normal coordinates

$$ds^2 = \Omega^2 d\hat{s}^2, \quad d\hat{s}^2 = -dn^2 + g_{ij}dx^i dx^j$$

- Hypersurface Σ: $\quad n = 0$
- Extrinsic curvature tensor:

$$\hat{K}_{ij} = -\frac{1}{2}\frac{\partial \hat{g}_{ij}}{\partial n}$$

- Spherical symmetry $\Rightarrow$

$$ds^2 = r^2 \left(\tilde{\gamma}_{ik} dx^i dx^k - (d\theta^2 + \sin^2\theta d\phi^2) \right)$$

$$d\hat{s}_2^2 = \tilde{\gamma}_{ik} dx^i dx^k = \tilde{\gamma}_{00}(\tau, n) d\tau^2 - dn^2$$

$\tilde{\gamma}_{00}(\tau, 0) = 1 \Rightarrow \tau$ — "conformal proper time" on the shell

$$\hat{K}_{00} = -\frac{1}{2}\frac{\partial \tilde{\gamma}_{00}}{\partial n} = \tilde{K}_{00}, \quad \tilde{K} = \tilde{K}_0^0 = -\frac{1}{2}\frac{\partial \log \tilde{\gamma}_{00}}{\partial n}$$

- $2 - dim$ scalar curvature: $\quad \tilde{R} = -2\tilde{K}_n + 2\tilde{K}^2$

I.6 Energy-momentum tensor

$$\delta S_{\text{matter}} \stackrel{def}{=} \frac{1}{2}\int T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\,dx$$

$$\delta S_{\text{matter}} \stackrel{def}{=} \frac{1}{2}\int \hat{T}_{\mu\nu}\sqrt{-\hat{g}}\,\delta \hat{g}^{\mu\nu}\,dx$$

$$\hat{T}_{\mu\nu} = r^2 T_{\mu\nu}, \quad \hat{T}^{\nu}_{\mu} = r^4 T^{\nu}_{\mu}, \quad \hat{T}^{\mu\nu} = r^6 T^{\mu\nu}$$

$$\boxed{\hat{T}_{\mu\nu} \stackrel{def}{=} \hat{S}_{\mu\nu}\delta(n) + [\hat{T}_{\mu\nu}]\Theta(n) + \hat{T}^{(-)}_{\mu\nu}}$$

$\hat{S}_{\mu\nu}$ — surface energy-momentum tensor
$\delta(n)$ — Dirac's δ–function
$\Theta(n)$ — Heaviside step function

$$\Theta(n) = \begin{cases} 1, & \text{if } n > 0 \ (+) \\ 0, & \text{if } n < 0 \ (-) \end{cases}$$

$\Theta^2 = \Theta, \ \ \Theta'(n) = \delta(n)$
$[\ldots] =$ "jump" $\ \Rightarrow\ [\hat{T}_{\mu\nu}] = [\hat{T}^{(+)}_{\mu\nu} - \hat{T}^{(-)}_{\mu\nu}]$

The End of Chapter I

(to be continued)

II. Quadratic Gravity

- A.D. Sakharov (1967)
- Induced gravity, conformal anomaly, particle creation (L. Parker, S. Fulling, Y.B. Zel'dovich, A.A. Starobinskii, ...).
- Lagrangian:

$$\left(\alpha R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2 - \frac{1}{16\pi G}(R - 2\Lambda)\right)\sqrt{-g}$$

- Singular hypersurface Σ ($n = 0$ in Gaussian normal coordinates),

$$T_{\mu\nu} = S_{\mu\nu}\delta(n) + [T_{\mu\nu}]\theta(n) + T^{(-)}_{\mu\nu}.$$

- General matching conditions – Senovilla et al.

• Main difference from GR:

Scalar curvature R may have only a jump at Σ (no term $\sim \delta(n) \Rightarrow \delta^2(n)$) in quadratic terms). The extrinsic curvature tensor

$$K_{\mu\nu} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial n}$$

must be continuous on Σ:

$$[K_{\mu\nu}] = 0.$$

In the presence of the thin shell in $T_{\mu\nu}$ ($S_{\mu\nu} \neq 0$) there is no smooth transition from a quadratic gravity to General Relativity. Instead – we may have the double layer ($\sim \delta'(n)$ term in the second derivative of R).

Because of the absence of the corresponding counterpart in $T_{\mu\nu}$ (no mass dipoles), the double layer is of pure geometric origin. It can be considered as the gravitational shock wave.

The End of Chapter II

(to be continued)

III. Weyl + Einstein gravity

- All the quadratic terms are combined in C^2.
- Motivations:

We are interested in the creation of universe "from nothing" (A. Vilenkin, ...) $\Rightarrow$ extra symmetry.
We are interesting in particle creation during the cosmological evolution $\Rightarrow$

$$(Nu^{\mu})_{;\mu} = \beta C^2$$

(fundamental result by Y.B. Zel'dovich and A.A. Starobinsky, 1977)

- Phenomenological description

$$S_{\mathrm{matter}} = S_{\mathrm{hydro}} + S_{\mathrm{creation}}$$

$$S_{\mathrm{hydro}} = -\int \varepsilon(N)\sqrt{-g}\,dx + \int \lambda_0(u^{\mu}u_{\mu} - 1)\sqrt{-g}\,dx + \int \lambda_2 X_{,\mu}u^{\mu}\sqrt{-g}\,dx$$

$$S_{\mathrm{creation}} = \int \lambda_1\left((Nu^{\mu})_{;\mu} - \beta C^2\right)\sqrt{-g}\,dx$$

If $\lambda_1 = +\alpha_0 + \tilde{\lambda}_1 \Longrightarrow$

$$-\int \frac{\alpha_0}{\beta}(Nu^\mu)_{;\mu}\sqrt{-g}dx \quad = \quad \text{surface integral}$$

That is, the Weyl term

$$-\alpha_0 \int C^2\sqrt{-g}dx \quad \text{is already there}$$

$$S_{\rm tot} = \int \beta\lambda_1 C^2\sqrt{-g}dx - \frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}dx + S_{\rm hydro}$$

- Field equations:

$$8\pi B_{\mu\nu}[\lambda_1] + \frac{1}{8\pi G}(G_{\mu\nu} - \Lambda g_{\mu\nu}) = T_{\mu\nu}$$

- Modified Bach tensor

$$B[\lambda_1] = (\lambda_1 C_{\mu\sigma\nu\lambda})^{;\lambda;\sigma} + \frac{1}{2}\lambda_1 C_{\mu\lambda\nu\sigma} R^{\lambda\sigma}$$

$G_{\mu\nu}$ – Einstein tensor
$g_{\mu\nu}$ – metric tensor

$$T_{\mu\nu} = T_{\mu\nu}^{\mathrm{hydro}} = (\varepsilon + p)u_\mu u_\nu - p g_{\mu\nu}$$

- Conformal transformation:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \Omega^2 d\hat{s}^2 = \Omega^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}; \quad g^{\mu\nu} = \frac{1}{\Omega^2}\hat{g}^{\mu\nu}$$

$$B_{\mu\nu}[\lambda_1] = \frac{1}{\Omega^2}\hat{B}_{\mu\nu}[\lambda_1]$$

$$T_{\mu\nu} \equiv \frac{1}{\Omega^2}\hat{T}_{\mu\nu}$$

- Spherical symmetry:

$$\Omega^2 = r^2$$

$$ds^2 = r^2 \left(\tilde{\gamma}_{ik} dx^i dx^k - (d\theta^2 + \sin^2\theta d\phi^2) \right)$$

$i = 0, 1$

$$d\tilde{s}_2^2 = \tilde{\gamma}_{ik} dx^i dx^k; \quad \tilde{R} \quad \text{– two-dim curvature}$$

- Gauss normal coordinate system:

$$d\tilde{s}_2^2 = \tilde{\gamma}_{00}(\tau, n) d\tau^2 - dn^2; \quad \tilde{\gamma}_{00}(\tau, 0) = 1.$$

$n = 0$ – time-like singular hypersurface Σ (just the world-line)

- Field equations: ($||$ – covariant derivative with respect $\tilde{\gamma}_{ik}$)

$$\frac{4}{3}\beta\left\{(\lambda_1(\tilde{R}-2))^{||p}_{||p}\tilde{\gamma}_{ik}-(\lambda_1(\tilde{R}-2))_{||ik}+\lambda_1\frac{\tilde{R}^2-4}{4}\tilde{\gamma}_{ik}\right\}$$

$$+\frac{1}{8\pi G}\left\{r^2(1+\Lambda r^2)\tilde{\gamma}_{ik}-2r(r_{||ik}-r^{||p}_{||p}\tilde{\gamma}_{ik})+4r_i r_k-r_p r^p\tilde{\gamma}_{ik}\right\}$$

$$=\tilde{T}_{ik}\quad i,k=0,1$$

(trace)

$$2-\tilde{R}+\frac{6r^{||p}_{||p}}{r}=4\Lambda r^2+\frac{8\pi G}{r^2}(\hat{T}^p_p+2\hat{T}^2_2)$$

$$\hat{T}_{\mu\nu}=\hat{S}_{\mu\nu}\delta(n)+[\hat{T}_{\mu\nu}]\theta(n)+\hat{T}^{(-)}_{\mu\nu}$$

$$[\ \ ]=(+)-(-)$$

(00)

$$\frac{4}{3}\beta\left\{-(\lambda_1(\tilde{R}-2))_{||nn}\tilde{\gamma}_{00}+\lambda_1\frac{\tilde{R}^2-4}{4}\right\}+$$

$$\frac{1}{8\pi G}\left\{r^2(1-\Lambda r^2)\tilde{\gamma}_{00}+3\dot{r}^2+r_n^2\tilde{\gamma}_{00}\right\}=\hat{T}_{00}$$

(0n)

$$-\frac{4}{3}\beta(\lambda_1(\tilde{R}-2))_{||0n}+\frac{1}{8\pi G}\left\{-2rr_{||0n}+4\dot{r}r_n\right\}=\hat{T}_{0n}$$

(nn)

$$-\frac{4}{3}\beta\left\{(\lambda_1(\tilde{R}-2))_{||00}\tilde{\gamma}^{00}+\lambda_1\frac{\tilde{R}^2-4}{4}\right\}-$$

$$-\frac{1}{8\pi G}\left\{r^2(1-\Lambda r^2)+2r\tilde{\gamma}^{00}r_{||00}-\tilde{\gamma}^{00}\dot{r}^2-3r_n^2\right\}=\hat{T}_{nn}$$

(trace)

$$2-\tilde{R}+\frac{6}{r}(\tilde{\gamma}^{00}r_{||00}-r_{||nn})=4\Lambda r^2+\frac{8\pi G}{r^2}(\hat{T}_0^0+\hat{T}_n^n+2\hat{T}_2^2)$$

- Extrinsic curvature:

$$\tilde{K}_{ij}=-\frac{1}{2}\tilde{\gamma}_{ij,n}\Rightarrow$$

$$\tilde{K}_{00}=-\frac{1}{2}\tilde{\gamma}_{00,n},\quad \tilde{K}=-\frac{1}{2}\tilde{\gamma}^{00}\tilde{\gamma}_{00,n}$$

$$\tilde{R}=-2\tilde{K}_{,n}+2\tilde{K}^2$$

The End of Chapter III

(to be continued)

IV. Double layer:

Appears if $[\tilde{R}] \neq 0$ $(n = 0) \Rightarrow$
$[\tilde{K}] = 0$
Only in (00) - equation

- Matching conditions:

$$\frac{4}{3}\beta\left\{-[(\lambda_1(\tilde{R}-2))_{,n}] + f(\tau)[\lambda_1(\tilde{R}-2)]\right\} - \frac{1}{4\pi G} r[r_n] = \hat{S}^0_0$$

$$\frac{4}{3}\beta[(\lambda_1(\tilde{R}-2))^{\cdot}] = \hat{S}^n_0$$

$$\frac{4}{3}\beta\tilde{K}[\lambda_1(\tilde{R}-2)] = \hat{S}^n_n$$

$$-\frac{3r}{4\pi G}[r_{,n}] = \hat{S}^0_0 + \hat{S}^n_n + 2\hat{S}^2_2$$

$\hat{S}^n_0$ - ?, $\hat{S}^n_n$ - ? (Senovilla)
($\hat{S}^n_0 = \hat{S}^n_n = 0$ in General Relativity).

$f(\tau)$ – arbitrary function (where from?)

$$\alpha_1\delta'(n) + \beta_1\delta(n) + \ldots = \beta_2\delta(n) + \ldots \quad \Rightarrow$$

$$\phi(\tau,n)\left(\alpha_1\delta'(n) + \beta_1\delta(n) + \ldots\right) = \phi(\tau,n)(\beta_2\delta(n) + \ldots)$$

$$\phi(\tau,0) \neq 0$$

$$-(\phi_{,n}\alpha_1 + \phi\alpha_{1,n} + \phi\beta_1) = \phi(\tau,0)\beta_2 \quad \Rightarrow$$

$$-\frac{\phi_{,n}}{\phi}\alpha_1 + \alpha_{1,n} + \beta_1 = \beta_2 \quad \Rightarrow f(\tau)$$

- But, it is not the end of the story:

$$(00) = \hat{T}_{00}$$

$$\binom{0}{0} = \tilde{\gamma}^{00}\hat{T}_{00} = \hat{T}^0_0$$

– extra term $\sim \tilde{\gamma}_{00,n} \Rightarrow \sim \tilde{K}$
to avoid such an ambiguity: $\tilde{K}(\tau,0) = 0$ (!!!)

- Final version:

$$2\beta\left\{-[(\lambda_1(\tilde{R}-2))_{,n}]+f(\tau)[\lambda_1(\tilde{R}-2)]\right\}=\hat{S}_0^0-\hat{S}_2^2$$

$$\frac{4}{3}\beta[(\lambda_1(\tilde{R}-2))^{\cdot}]=\hat{S}_0^n$$

$$\hat{S}_n^n=0;\quad \tilde{K}=0$$

$$-\frac{3r}{4\pi G}[r_{,n}]=\hat{S}_0^0+2\hat{S}_2^2$$

- Continuity equation:

$$\dot{\hat{S}}_0^0-\frac{4}{3}\beta\left(f(\tau)[\lambda_1(\tilde{R}-2)]\right)^{\cdot}-\frac{\dot{r}}{r}(\hat{S}_0^0+2\hat{S}_2^2)+[T_0^n]=0$$

The End of Chapter IV

(to be continued)

V. Toy model

- Universe is created from nothing.

Empty and with maximal symmetry.
Homogeneous and isotropic apace-time $\Rightarrow$
Weyl tensor is zero, $C_{\mu\nu\lambda\sigma} = 0 \Rightarrow \tilde{R} = 2$.
Conformal to the (anti)de Sitter manifold.

- Subsequent evolution - fluctuations with $\tilde{R} \neq 2 \Rightarrow$ particle creation.
- No thin shells ($\sim \delta(n)$) – only double layer ($\sim \delta'(n)$) at the boundary $n = 0$.
- Matching conditions become

$$2\beta \left\{ -[(\lambda_1(\tilde{R} - 2))_{,n}] + f(\tau)[\lambda_1(\tilde{R} - 2)] \right\} = 0$$

$$\frac{4}{3}\beta[(\lambda_1(\tilde{R} - 2))^{\cdot}] = 0$$

$$f(\tau) = f_0 = const$$

$$\tilde{K} = 0$$

- Further simplification: pure Weyl gravity ($G \to \infty$)
- The view from inside ($n \leq 0$)
- Energy-momentum tensor is necessarily traceless $\Rightarrow \varepsilon = 3p$,

$$\varepsilon = \mu_0 N^{4/3}$$

$$Y = Nr^3$$

N – invariant particle number density,
r – radius
At the boundary ($n = 0 - 0$)

$$\lambda_1(\tilde{R} - 2) = A = const \quad \text{matching condition}$$

$$Y^{1/3} = -\frac{3}{4\mu_0}\frac{d\lambda_1}{d\tau} \quad \text{hydrodynamics}$$

$$Y^{4/3} = -\frac{\beta}{\mu_0}A(\tilde{R} + 2) \quad \text{Bach equations}$$

$$\frac{d\tilde{R}}{d\tau} = \frac{4\mu_0(\tilde{R}-2)^2}{3A}\left(-\frac{\beta}{\mu_0}A(\tilde{R}+2)\right)^{1/4}$$

$$\lambda_1 = \frac{A}{\tilde{R}-2}$$

$$Y^{4/3} = -\frac{\beta}{\mu_0}A(\tilde{R}+2)$$

- Three types of solutions:

1. $\tilde{R} < -2 \Rightarrow \frac{\beta}{\mu_0}A > 0 \Rightarrow d\tilde{R}/d\tau > 0 \Rightarrow$
$Y \to 0 \Rightarrow r \to 0$

2. $-2 < \tilde{R} < 2 \Rightarrow \frac{\beta}{\mu_0}A < 0 \Rightarrow d\tilde{R}/d\tau < 0 \Rightarrow$
$Y \to 0 \Rightarrow r \to 0$

Punctured vacuum !

3. $\tilde{R} > 2 \Rightarrow \frac{\beta}{\mu_0}A < 0 \Rightarrow d\tilde{R}/d\tau < 0 \Rightarrow \tilde{R} \to +2$
$Y \to const$

Isotropization (Y.B. Zel'dovich)

The End of Chapter V

(to be continued)

The End

Thanks to all