Cosmological perturbations during the kinetic inflation in the Horndeski theory

Sergey Sushkov
KAZAN FEDERAL UNIVERSITY

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Plan of the talk

- Scalar fields in gravitational physics
- Horndeski model
- Cosmological models with nonminimal derivative coupling
  - No potential
  - Cosmological constant
  - Power-law potential
- The screening Horndeski cosmologies
- Perturbations
- Summary
Scalar fields in gravitational physics:

- gravitational potential in Newtonian gravity
- variation of “fundamental” constants
- Brans-Dicke theory initially elaborated to solve the Mach problem
- various compactification schemes
- the low-energy limit of the superstring theory
- scalar field as inflaton
- scalar field as dark energy and/or dark matter
- fundamental Higgs bosons, neutrinos, axions, . . .
- etc. . .
In 1974, Horndeski derived the action of the most general scalar-tensor theories with second-order equations of motion

[\text{G. Horndeski, Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space, IJTP 10, 363 (1974)}]

Horndeski Lagrangian:

\[
L_H = \sqrt{-g} \left( \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right)
\]

\[
\mathcal{L}_2 = G_2(X, \phi), \quad \mathcal{L}_3 = G_3(X, \phi) \Box \phi, \quad \mathcal{L}_4 = G_4(X, \phi) R + \partial_X G_4(X, \phi) \delta^\mu_\alpha \delta^\nu_\beta \nabla^\alpha \phi \nabla^\beta \phi, \quad \mathcal{L}_5 = G_5(X, \phi) G_{\mu\nu} \nabla^{\mu\nu} \phi - \frac{1}{6} \partial_X G_5(X, \phi) \delta^\mu_\alpha \delta^\nu_\beta \nabla^\alpha \phi \nabla^\beta \phi \nabla^\gamma \phi,
\]

where \( X = -\frac{1}{2} (\nabla \phi)^2 \), and \( G_k(X, \phi) \) are arbitrary functions,

and \( \delta^{\lambda \rho}_{\nu \alpha} = 2! \delta^{\lambda}_{[\nu} \delta^{\rho}_{\alpha]}, \quad \delta^\lambda_{\nu \alpha \beta} = 3! \delta^{\lambda}_{[\nu} \delta^{\rho}_{\alpha} \delta^\sigma_{\beta]} \)
There is a special subclass of the theory, sometimes called Fab Four (F4), for which the coefficients are chosen such that the Lagrangian becomes

$$L_{F4} = \sqrt{-g} (\mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda)$$

with

$$\mathcal{L}_J = V_J(\phi) G_{\mu\nu} \nabla^{\mu}\phi \nabla^{\nu}\phi,$$

$$\mathcal{L}_P = V_P(\phi) P_{\mu\nu\rho\sigma} \nabla^{\mu}\phi \nabla^{\rho}\phi \nabla^{\nu\sigma}\phi,$$

$$\mathcal{L}_G = V_G(\phi) R,$$

$$\mathcal{L}_R = V_R(\phi) (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2).$$

Here the double dual of the Riemann tensor is

$$P^{\mu\nu}_{\alpha\beta} = -\frac{1}{4} \delta^{\mu\nu\gamma\delta} R^{\sigma\lambda}_{\gamma\delta} = -R^{\mu\nu}_{\alpha\beta} + 2R^{\mu}_{[\alpha} \delta^{\nu}_{\beta]} - 2R^{\nu}_{[\alpha} \delta^{\mu}_{\beta]} - R\delta^{\mu}_{[\alpha} \delta^{\nu}_{\beta]},$$

whose contraction is the Einstein tensor, $P^\mu_{\nu\alpha} = G^\mu_{\nu}.$
Fab Four subclass of the Horndeski theory

**Fab Four Lagrangian:**

\[ L_{F4} = \sqrt{-g} \left( \mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda \right) \]

- The Fab Four model is distinguished by the *screening property* – it is the most general subclass of the Horndeski theory in which flat space is a solution, despite the presence of the cosmological term \( \Lambda \).
- This property suggests that \( \Lambda \) is actually irrelevant and hence there is no need to explain its value.
- Indeed, however large \( \Lambda \) is, Minkowski space is always a solution and so one may hope that a slowly accelerating universe will be a solution as well.
Theory with nonminimal kinetic coupling

Action:

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{Pl}^2 R - (\varepsilon g_{\mu\nu} + \eta G_{\mu\nu}) \nabla^\mu \phi \nabla_\nu \phi - 2V(\phi) \right] + S_m \]

Field equations:

\[ M_{Pl}^2 G_{\mu\nu} = T^{(\phi)}_{\mu\nu} + \eta \Theta_{\mu\nu} + T^{(m)}_{\mu\nu} \]

\[ [\varepsilon g^{\mu\nu} + \eta G^{\mu\nu}] \nabla_\mu \nabla_\nu \phi = V'_\phi \]

\[ T^{(\phi)}_{\mu\nu} = \varepsilon \left[ \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right] - g_{\mu\nu} V(\phi), \]

\[ \Theta_{\mu\nu} = -\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R + 2 \nabla_\alpha \phi \nabla_{(\mu} \phi R^{\alpha}_{\nu)} - \frac{1}{2} (\nabla \phi)^2 G_{\mu\nu} + \nabla^\alpha \phi \nabla^\beta \phi R_{\mu\alpha\nu\beta} \]

\[ + \nabla_\mu \nabla^\alpha \phi \nabla_\nu \nabla_\alpha \phi - \nabla_\mu \nabla_\nu \phi \Box \phi + g_{\mu\nu} \left[ -\frac{1}{2} \nabla^\alpha \phi \nabla^\beta \phi \nabla_\alpha \nabla_\beta \phi + \frac{1}{2} (\Box \phi)^2 \right] \]

\[ - \nabla_\alpha \phi \nabla_\beta \phi R^{\alpha\beta} \]

\[ T^{(m)}_{\mu\nu} = (\rho + p) U_\mu U_\nu + pg_{\mu\nu}, \]

Notice: The field equations are of second order!
Cosmological models: General formulas

Ansatz:

\[ ds^2 = -dt^2 + a^2(t)dx^2, \]
\[ \phi = \phi(t) \]

\( a(t) \) cosmological factor, \( H = \dot{a}/a \) Hubble parameter

Field equations:

\[ 3M_{\text{Pl}}^2 H^2 = \frac{1}{2} \dot{\phi}^2 (\epsilon - 9\eta H^2) + V(\phi), \]
\[ M_{\text{Pl}}^2 (2\dot{H} + 3H^2) = -\frac{1}{2} \dot{\phi}^2 \left[ \epsilon + \eta \left( 2\dot{H} + 3H^2 + 4H \ddot{\phi} \dot{\phi}^{-1} \right) \right] + V(\phi), \]
\[ \frac{d}{dt} \left[ (\epsilon - 3\eta H^2) a^3 \dot{\phi} \right] = -a^3 \frac{dV(\phi)}{d\phi} \]
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\[ \frac{d}{dt} \left[ (\epsilon - 3\eta H^2)a^3 \dot{\phi} \right] = -a^3 \frac{dV(\phi)}{d\phi} \]

\[ V(\phi) \equiv \text{const} \implies \dot{\phi} = \frac{Q}{a^3(\epsilon - 3\eta H^2)} \quad Q \text{ is a scalar charge} \]
Trivial model without kinetic coupling, i.e. $\eta = 0$

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{\text{Pl}}^2 R - (\nabla \phi)^2 \right]
\]
Trivial model without kinetic coupling, i.e. $\eta = 0$

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{\text{Pl}}^2 R - (\nabla \phi)^2 \right] \]

Solution:

\[ a_0(t) = t^{1/3}; \quad \phi_0(t) = \frac{1}{2\sqrt{3\pi}} \ln t \]

\[ ds_0^2 = -dt^2 + t^{2/3} \, dx^2 \]

$t = 0$ is an initial singularity
Model without free kinetic term, i.e. $\epsilon = 0$

$$S = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ M_{Pl}^2 R - \eta G^{\mu \nu} \phi_{,\mu} \phi_{,\nu} \right]$$
Model without free kinetic term, i.e. $\epsilon = 0$

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{Pl}^2 R - \eta G^{\mu\nu} \phi,_{\mu} \phi,_{\nu} \right] \]

Solution:

\[ a(t) = t^{2/3}; \quad \phi(t) = \frac{t}{2\sqrt{3\pi|\eta|}}, \quad \eta < 0 \]

\[ ds_0^2 = -dt^2 + t^{4/3} dx^2 \]

\[ t = 0 \text{ is an initial singularity} \]
Model for an ordinary scalar field ($\epsilon = 1$) with nonminimal kinetic coupling $\eta \neq 0$

$$S = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ M_{\text{Pl}}^2 R - (g^{\mu\nu} + \eta G^{\mu\nu}) \phi_{,\mu} \phi_{,\nu} \right]$$
Model for an ordinary scalar field ($\epsilon = 1$) with nonminimal kinetic coupling $\eta \neq 0$

$$S = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ M_{Pl}^2 R - (g^{\mu\nu} + \eta G^{\mu\nu}) \phi,_{\mu} \phi,_{\nu} \right]$$

Asymptotic for $t \to \infty$:

$$a(t) \sim a_0(t) = t^{1/3}; \quad \phi(t) \sim \phi_0(t) = \frac{1}{2\sqrt{3\pi}} \ln t$$

Notice: At large times the model with $\eta \neq 0$ has the same behavior like that with $\eta = 0$
Asymptotics for early times

The case $\eta < 0$:

$$a_{t \to 0} \approx t^{2/3}, \quad \phi_{t \to 0} \approx \frac{t}{2 \sqrt{3\pi |\eta|}}$$

$$ds^2_{t \to 0} = -dt^2 + t^{4/3} dx^2$$

$t = 0$ is an initial singularity

The case $\eta > 0$:

$$a_{t \to -\infty} \approx e^{H_\eta t}, \quad \phi_{t \to -\infty} \approx Ce^{-t / \sqrt{\eta}}$$

$$ds^2_{t \to -\infty} = -dt^2 + e^{2H_\eta t} dx^2$$

de Sitter asymptotic with $H_\eta = 1 / \sqrt{9\eta}$
Cosmological models: III. No potential $V(\phi) \equiv 0$

**Plots of $\alpha = \ln a$ in case $\eta \neq 0$, $\epsilon = 1$, $V = 0$.**

(a) $\eta < 0$; 
$\eta = 0; -1; -10; -100$

(b) $\eta > 0$; 
$\eta = 0; 1; 10; 100$

**De Sitter asymptotics:** $\alpha(t) = \frac{t}{\sqrt{9\eta}} \ \Rightarrow \ H = \frac{1}{\sqrt{9\eta}}$

**Notice:** In the model with nonminimal kinetic coupling one get de Sitter phase without any potential!
Models with the constant potential $V(\phi) = M_{Pl}^2\Lambda = \text{const}$

$$S = \int d^4x \sqrt{-g} \left[ M_{Pl}^2 (R - 2\Lambda) - [\epsilon g^{\mu\nu} + \eta G^{\mu\nu}] \phi,_{\mu} \phi,_{\nu} \right]$$
Models with the constant potential $V(\phi) = M_{Pl}^2 \Lambda = \text{const}$

$$S = \int d^4 x \sqrt{-g} \left[ M_{Pl}^2 (R - 2\Lambda) - [\epsilon g^{\mu\nu} + \eta G^{\mu\nu}] \phi,_{\mu} \phi,_{\nu} \right]$$

There are two exact de Sitter solutions:

I. $\alpha(t) = H_{\Lambda} t, \quad \phi(t) = \phi_0 = \text{const}$,

II. $\alpha(t) = \frac{t}{\sqrt{3|\eta|}}, \quad \phi(t) = M_{Pl} \left| \frac{3\eta H_{\Lambda}^2 - 1}{\eta} \right|^{1/2} t$,

$$H_{\Lambda} = \sqrt{\Lambda/3}$$
Plots of $\alpha(t)$ in case $\eta > 0$, $\epsilon = 1$, $V = M_P^2 \Lambda$

(a) $H_\Lambda^2 < \dot{\alpha}^2 < 1/9\eta$

(b) $1/9\eta < \dot{\alpha}^2 < 1/3\eta < H_\Lambda^2$

De Sitter asymptotics:

$\alpha_1(t) = H_\Lambda t$ (dashed),

$\alpha_2(t) = t/\sqrt{9\eta}$ (dash-dotted),

$\alpha_3(t) = t/\sqrt{3\eta}$ (dotted).
Role of potential

\[ S = \int d^4x \sqrt{-g} \left\{ M_{Pl}^2 R - [g^{\mu\nu} + \eta G^{\mu\nu}] \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - 2V(\phi) \right\} \]

What a role does a potential play in cosmological models with the nonminimal kinetic coupling?
Models with the quadratic potential $V(\phi) = \frac{1}{2} m^2 \phi^2$

Primary (early-time) “kinetic” inflation:

$$H_{t \to -\infty} \approx \frac{1}{\sqrt{9\eta}} (1 + \frac{1}{2} \eta m^2)$$

Late-time cosmological scenarios:
Oscillatory asymptotic or “graceful” exit from inflation

$$H_{t \to \infty} \approx \frac{2}{3t} \left[ 1 - \frac{\sin 2mt}{2mt} \right]$$

quasi-de Sitter asymptotic or secondary inflation

$$H_{t \to \infty} \approx \frac{1}{\sqrt{3\eta}} \left( 1 \pm \sqrt{\frac{1}{6} \eta m^2} \right)$$
Cosmological models: Power-law potential

Initial conditions
\[ \phi_0 = \dot{\phi}_0 \]

De Sitter asymptotics: \[ H_{t \to -\infty} \approx 1/\sqrt{9\eta}(1 + \frac{1}{2}\eta m^2), \]
\[ H_{t \to \infty} \approx 1/\sqrt{3\eta}\left(1 \mp \sqrt{\frac{1}{6}\eta m^2}\right). \]
The FLRW ansatz for the metric:

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - K r^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right], \]

\( a(t) \) cosmolocial factor, \( H = \dot{a}/a \) Hubble parameter

Gravitational equations:

\[-3M_{\text{Pl}}^2 \left( H^2 + \frac{K}{a^2} \right) + \frac{1}{2} \varepsilon \psi^2 - \frac{3}{2} \eta \psi^2 \left( 3H^2 + \frac{K}{a^2} \right) + \Lambda + \rho = 0,\]

\[-M_{\text{Pl}}^2 \left( 2\dot{H} + 3H^2 + \frac{K}{a^2} \right) - \frac{1}{2} \varepsilon \psi^2 - \eta \psi^2 \left( \dot{H} + \frac{3}{2} H^2 - \frac{K}{a^2} + 2H \frac{\dot{\psi}}{\psi} \right) + \Lambda - p = 0.\]

The scalar field equation:

\[ \frac{1}{a^3} \frac{d}{dt} \left( a^3 \left( 3\eta \left( H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi \right) = 0,\]

where \( \psi = \dot{\phi} \), and \( \phi = \phi(t) \) is a homogeneous scalar field.
The first integral of the scalar field equation:

\[ a^3 \left( 3\eta \left( H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi = Q, \]

where \( Q \) is the Noether charge associated with the shift symmetry \( \phi \rightarrow \phi + \phi_0 \).

Let \( Q = 0 \). One finds in this case two different solutions:

**GR branch:** \( \psi = 0 \quad \Rightarrow \quad H^2 + \frac{K}{a^2} = \frac{\Lambda + \rho}{3M_{\text{Pl}}^2} \)

**Screening branch:** \( H^2 + \frac{K}{a^2} = \frac{\varepsilon}{3\eta} \quad \Rightarrow \quad \psi^2 = \frac{\eta (\Lambda + \rho) - \varepsilon M_{\text{Pl}}^2}{\eta (\varepsilon - 3\eta K/a^2)} \)

**NOTICE:** The role of the cosmological constant in the screening solution is played by \( \varepsilon/3\eta \) while the \( \Lambda \)-term is screened and makes no contribution to the universe acceleration.

Note also that the matter density \( \rho \) is screened in the same sense.
Let \( Q \neq 0 \), then

\[
\psi = \frac{Q}{a^3 \left[ 3\eta (H^2 + \frac{K}{a^2}) - \varepsilon \right]},
\]

and the modified Friedmann equation reads

\[
3M_{Pl}^2 \left( H^2 + \frac{K}{a^2} \right) = \frac{Q^2 \left[ \varepsilon - 3\eta (3H^2 + \frac{K}{a^2}) \right]}{2a^6 \left[ \varepsilon - 3\eta (H^2 + \frac{K}{a^2}) \right]^2} + \Lambda + \rho.
\]

Introducing dimensionless values and density parameters

\[
H^2 = H_0^2 y, \quad a = a_0 a, \quad \rho_{cr} = 3M_{Pl}^2 H_0^2, \quad \eta = \frac{\varepsilon}{3\eta H_0^2},
\]

\[
\Omega_0 = \frac{\Lambda}{\rho_{cr}}, \quad \Omega_2 = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_6 = \frac{Q^2}{6\eta a_0^6 H_0^2 \rho_{cr}}, \quad \rho = \rho_{cr} \left( \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} \right)
\]

gives

the master equation:

\[
y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6 \left[ \eta - 3y + \frac{\Omega_2}{a^2} \right]}{a^6 \left[ \eta - y + \frac{\Omega_2}{a^2} \right]^2}
\]
Asymptotical behavior: Late time limit $a \to \infty$

**GR branch:**

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{(\eta - 3\Omega_0)\Omega_6}{(\Omega_0 - \eta)^2 a^6} + \mathcal{O}\left(\frac{1}{a^7}\right) \implies H^2 \to \Lambda/3$$

**Notice:** The GR solution is stable (no ghost) if and only if $\eta > \Omega_0$.

**Screening branches:**

$$y_{\pm} = \eta + \frac{\Omega_2}{a^2} \pm \frac{\chi}{(\Omega_0 - \eta)} \frac{\Omega_3}{a^3} \pm \frac{\Omega_2 \Omega_6}{\chi a^5} - \frac{\Omega_6 (\eta - 3\Omega_0)}{2(\Omega_0 - \eta)^2 a^6} \pm \Omega_3 \chi + \mathcal{O}\left(\frac{1}{a^7}\right)$$

$$\implies H^2 \to \varepsilon/3\alpha$$

**Notice:** The screening solutions are stable (no ghost) if and only if $0 < \eta < \Omega_0$. 
Asymptotical behavior: The limit $a \to 0$

**GR branch:**

\[
y = \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} + \frac{\Omega_2 \Omega_4 - 3 \Omega_6}{\Omega_4 a^2} + \frac{3 \Omega_3 \Omega_6}{\Omega_4 a} + O(1)
\]

**Notice:** The GR solution is unstable

**Screening branch:**

\[
y_+ = \frac{3 \Omega_6}{\Omega_4 a^2} - \frac{3 \Omega_3 \Omega_6}{\Omega_4^2 a} + \frac{5}{3} \eta + \frac{3 \Omega_6 \Omega_3^2 + 9 \Omega_6^2}{\Omega_4^3} + O(a),
\]

\[
y_- = \frac{1}{\sqrt{9\eta}} + \frac{4 \eta^2}{27 \Omega_6} \left( \Omega_4 a^2 + \Omega_3 a^3 \right) + O(a^4)
\]

**Notice:** Both screening solutions are stable
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \eta - 3y + \frac{\Omega_2}{a^2} \right] \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \ \Omega_2 = 0, \ \Omega_3 = \Omega_4 = 0 \) and for \( \eta = 6 \)
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \eta - 3y + \frac{\Omega_2}{a^2} \right] \left[ \eta - y + \frac{\Omega_2}{a^2} \right]^2 \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \Omega_2 = 0, \Omega_3 = \Omega_4 = 0, \eta = 0.2 \)
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \eta - 3y + \frac{\Omega_2}{a^2} \right] \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \ \Omega_3 = 5, \ \Omega_4 = 0, \ \eta = 0.2 \). One has \( \Omega_2 = 0 \).
The nonminimal kinetic coupling provides an essentially new inflationary mechanism which does not need any fine-tuned potential.

At early cosmological times the coupling $\eta$-terms in the field equations are dominating and provide the quasi-De Sitter behavior of the scale factor: $a(t) \propto e^{H_\eta t}$ with $H_\eta = 1/\sqrt{9\eta}$.

The model provides a natural mechanism of epoch change without any fine-tuned potential.

The nonminimal kinetic coupling crucially changes a role of the scalar potential. Power-law and Higgs-like potentials with kinetic coupling provide accelerated regimes of the Universe evolution.
Scalar perturbations (Newtonian gauge):

\[ ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)\delta_{ij}dx^i dx^j, \]

\[ \phi = \phi_0 + \delta\phi = \phi_0(1 + \varphi), \]

\[ \Psi(t, x) \ll 1, \Phi(t, x) \ll 1, \varphi(t, x) \ll 1 \]

Fourier transformations: \( \Psi(t, x) = \int dk e^{ikx}\Psi(t, k) \) and so on

Scalar modes:

\[ -3H(\dot{\Psi} - H\Phi) - \frac{k^2}{a^2}\Psi = 4\pi \left[ \ddot{\phi}^2\Phi - \dot{\phi}\dot{\delta}\phi \right. \]

\[ + \eta \left( 9H\dot{\phi}^2\dot{\Psi} - 18H^2\dot{\phi}^2\Phi + \frac{k^2}{a^2}\dot{\phi}^2\Psi + 9H^2\dot{\phi}\dot{\delta}\phi + 2\frac{k^2}{a^2}H\dot{\phi}\dot{\delta}\phi \right), \]

\[ \dot{\Psi} - H\Phi = 4\pi \left[ -\dot{\phi}\delta\phi + \eta \left( 3H\dot{\phi}^2\Phi - \dot{\phi}^2\dot{\Psi} - 2H\dot{\phi}\dot{\delta}\phi + 3H^2\dot{\phi}\dot{\delta}\phi \right) \right], \]

\[ \Phi + \Psi = -4\pi\eta \left[ \dot{\phi}^2(\Phi - \Psi) + 2(\ddot{\phi} + H\dot{\phi})\delta\phi \right] \]
Perturbations

Scalar perturbations (Newtonian gauge):

\[ ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)\delta_{ij}dx^i dx^j, \]
\[ \phi = \phi_0 + \delta \phi = \phi_0(1 + \varphi), \]
\[ \Psi(t, x) \ll 1, \Phi(t, x) \ll 1, \varphi(t, x) \ll 1 \]

Fourier transformations: \( \Psi(t, x) = \int dke^{ikx}\Psi(t,k) \) and so on

Scalar modes:

\[ -3H(\dot{\Psi} - H\Phi) - \frac{k^2}{a^2}\Psi = 4\pi \left[ \dot{\phi}^2 \Phi - \ddot{\phi} \dot{\phi} \delta \phi \right. \]
\[ + \eta \left( 9H\dot{\phi}^2 \dot{\Psi} - 18H^2 \dot{\phi}^2 \Phi + \frac{k^2}{a^2} \dot{\phi}^2 \Psi + 9H^2 \dot{\phi} \delta \phi + 2\frac{k^2}{a^2} H \dot{\phi} \delta \phi \right) \],
\[ \dot{\Psi} - H\Phi = 4\pi \left[ -\dot{\phi} \delta \phi + \eta \left( 3H \dot{\phi}^2 \Phi - \dot{\phi}^2 \dot{\Psi} - 2H \dot{\phi} \delta \phi + 3H^2 \dot{\phi} \delta \phi \right) \right], \]
\[ \Phi + \Psi = -4\pi \eta \left[ \dot{\phi}^2 (\Phi - \Psi) + 2(\ddot{\phi} + H \dot{\phi}) \delta \phi \right] \]

Notice: \( \Psi = -\Phi \) if \( \eta = 0 \), but generally \( \Psi \neq -\Phi \)!
On the inflationary stage at $t \to -\infty$ the unperturbed solutions are

$$a(t) = a_i e^{H_\eta(t-t_i)}, \quad \phi(t) = \phi_i e^{-3H_\eta(t-t_i)}, \quad H_\eta = \frac{1}{\sqrt{9\eta}}$$

**Scalar perturbations on the inflationary stage**

$$\dot{\Psi} = H_\eta \Phi - \frac{1}{12H_\eta} \frac{k^2}{a^2} (7\Psi + 3\Phi),$$

$$\dot{\Phi} = -H_\eta (6\Psi + 7\Phi) + \frac{1}{4H_\eta} \frac{k^2}{a^2} (7\Psi + 3\Phi). \quad a = a_i e^{H_\eta(t-t_i)}$$
Limiting cases:

A. $k/a \ll H_\eta$ (modes outside the Hubble horizon)

Scalar perturbations of metric:

$$\dot{\Psi} = H_\eta \Phi - \frac{1}{12 H_\eta} \frac{k^2}{a^2} (7 \Psi + 3 \Phi),$$

$$\dot{\Phi} = -H_\eta (6 \Psi + 7 \Phi) + \frac{1}{4 H_\eta} \frac{k^2}{a^2} (7 \Psi + 3 \Phi).$$

$$a = a_i e^{H_\eta (t-t_i)}$$

$$\Psi = \frac{1}{5} (6 \Psi_i + \Phi_i) e^{-H_\eta (t-t_i)} - \frac{1}{5} (\Psi_i + \Phi_i) e^{-6H_\eta (t-t_i)},$$

$$\Phi = -\frac{1}{5} (6 \Psi_i + \Phi_i) e^{-H_\eta (t-t_i)} + \frac{6}{5} (\Psi_i + \Phi_i) e^{-6H_\eta (t-t_i)},$$

$$\Psi_i = \Psi(t_i) \ll 1, \quad \Phi_i = \Phi(t_i) \ll 1, \quad t = t_i - \text{beginning of inflation}$$

Perturbs in course of inflation $t > t_i$: \(\Psi = -\Phi \sim e^{-H_\eta t} \sim a^{-1}\)

**NOTICE:** Scalar modes $k/a \ll H_\eta$ are exponentially decaying!
Perturbations in the inflationary epoch

B. $k/a \gg H_\eta$ (modes inside the Hubble horizon)

Scalar perturbations of metric:

\[
\dot{\Psi} = H_\eta \Phi - \frac{1}{12H_\eta} \frac{k^2}{a^2} (7\Psi + 3\Phi),
\]
\[
\dot{\Phi} = -H_\eta (6\Psi + 7\Phi) + \frac{1}{4H_\eta} \frac{k^2}{a^2} (7\Psi + 3\Phi). \quad a = a_i e^{H_\eta(t-t_i)}
\]

\[
\Psi = \frac{3}{2} (3\Psi_i + \Phi_i) - \frac{3}{2} \left( \frac{7}{3} \Psi_i + \Phi_i \right) \exp \left[ \frac{1}{12} \left( \frac{k}{H_\eta} \right)^2 \left( \frac{1}{a_i^2} - \frac{1}{a^2} \right) \right],
\]
\[
\Phi = -\frac{7}{2} (3\Psi_i + \Phi_i) + \frac{9}{2} \left( \frac{7}{3} \Psi_i + \Phi_i \right) \exp \left[ \frac{1}{12} \left( \frac{k}{H_\eta} \right)^2 \left( \frac{1}{a_i^2} - \frac{1}{a^2} \right) \right],
\]

Perturbs in course of inflation $t > t_i$ ($1/a_i^2 \gg 1/a^2$):

\[
\Psi, \Phi \to \exp \left[ \frac{1}{12} \left( \frac{k}{a_i H_\eta} \right)^2 \right] \gg 1
\]

NOTICE: Scalar modes $k/a \gg H_\eta$ are growing!
**TENDENCY:** During the inflation, modes with short wavelength are stretching and come beyond the Hubble horizon. After they have gone outside the Hubble horizon, they are exponentially decaying.

*Examples of numerical analysis for scalar mode evolution:*
Tensor perturbations:

\[ ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \]

\[ \partial_i h_{ij} = 0, \ h_{ii} = 0. \]

**Two polarizations:** \( h_{ij} \rightarrow h^+, \ h^\times \)

**Equation for tensor modes**

\[ (1 + 4\pi\eta\dot{\phi}^2)\ddot{h} + \left(3H + 4\pi\eta(2\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2)\right)\dot{h} + \frac{k^2}{a^2} (1 - 4\pi\eta\dot{\phi}^2)h = 0 \]
Tensor perturbations during the kinetic inflation

Tensor perturbation on the inflationary stage:

\[
\left(1 + 4\pi \phi_i^2 e^{-6H_\eta(t-t_i)}\right) \ddot{h} + 3H_\eta \left(1 - 4\pi \phi_i^2 e^{-6H_\eta(t-t_i)}\right) \dot{h} + \frac{k^2}{a^2} \left(1 - 4\pi \phi_i^2 e^{-6H_\eta(t-t_i)}\right) h = 0
\]

\[a(t) = a_i e^{H_\eta(t-t_i)}, \quad \phi(t) = \phi_i e^{-3H_\eta(t-t_i)}\]

The case \(4\pi \phi_i^2 \ll 1\):

\[\ddot{h} + 3H_\eta \dot{h} + \frac{k^2}{a^2} h = 0\]

A. \(k/a \ll H_\eta\) (outside the Hubble horizon) \(\Rightarrow\) constant modes

B. \(k/a \gg H_\eta\) (inside the Hubble horizon) \(\Rightarrow\) damping oscillating modes
Tensor perturbations during the kinetic inflation

The case $4\pi \phi_i^2 \gg 1$:  

$$\ddot{h} - 3H_\eta \dot{h} - \frac{k^2}{a^2} h = 0$$

A. $k/a \ll H_\eta$  
**modes outside the Hubble horizon**

$$\ddot{h} - 3H_\eta \dot{h} = 0 \quad \Rightarrow \quad h \propto e^{3H_\eta t} \quad \Rightarrow \quad \text{exponentially growing!}$$

B. $k/a \gg H_\eta$  
**modes inside the Hubble horizon**

$$\ddot{h} - \frac{k^2}{a^2} h = 0 \quad \Rightarrow \quad h \propto e^{\pm k e^{-H_\eta t}/H_\eta} \quad \Rightarrow \quad \text{constant modes}$$
Final conclusions

- Long-wave scalar modes $k/a \ll H_\eta$ are exponentially decaying during the kinetic inflation. Therefore, the large-scale structure of the Universe keeps to be homogeneous and isotropic.

- Short-wave scalar modes $k/a \gg H_\eta$ are growing during the narrow time interval when $k/a \approx H_\eta$. At this moment seeds for the Universe structure (clusters, galaxies, etc) could be formed. However, this is a regime of nonlinear perturbations, and hence one needs a nonperturbative analysis.
THANKS FOR YOUR ATTENTION!