De Sitter and Power-law Solutions in Non-local Gauss-Bonnet Gravity

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based on the work E. Elizalde, S.D. Odintsov, E.O. Pozdeeva, S.Yu. Vernov, arXiv:1805.10810, International Journal of Geometric Methods in Modern Physics Vol. 15 (2018) 1850188

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- inflation,
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Initial inflation and dark energy domination are both characterized by an accelerated expansion of the Universe with almost constant Hubble parameter H.
The other epochs of the Universe evolution are described by power-law solutions with $H = J/t$, where J is a positive constant.
In General Relativity, power-law solutions with $H = J/t$ correspond to models with a perfect fluid whose EoS parameter reads $w_{\mathrm{m}} = -1 + 2/(3J)$.
The radiation dominated epoch corresponds to solutions with $J = 1/2$, whereas the matter dominated one corresponds to $J = 2/3$.
When one addresses the issue to consider new modified gravity models, it is therefore important to check for the existence of de Sitter and power-law solutions in the discussed models.

MODIFIED GRAVITY MODELS

There are two basic motivations which lead cosmologists to modify gravity. The first one is an attempt to connect gravity with quantum physics, at least in a perturbative way, by including quantum correction terms to Einstein's equations. The second is an interest to describe the Universe evolution in a more natural way, without the dark energy and the dark matter components, which turn out to be avoidable in the modified models.

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There are lots of ways to deviate from Einstein's gravity:

- $F(R)$ gravity
- Addition of higher-derivative terms to the Einstein–Hilbert action
- Non-local gravity

Non-local models with inverse d'Alembertian

The following class of nonlocal gravity models has been proposed to explain current cosmic acceleration without dark energy[1]:

$$S_2 = \int d^4x\sqrt{-g}\left\{\frac{{M_{\mathrm{Pl}}}^2}{2}\left[R\left(1+f(\Box^{-1}R)\right)-2\Lambda\right]+\mathcal{L}_{\mathrm{m}}\right\} \qquad (1)$$

Here f is a differentiable function, Λ is the cosmological constant, $\mathcal{L}_{\mathrm{m}}$ is the matter Lagrangian, $\Box$ is covariant d'Alembertian for a scalar field.

$$\Box A \equiv \frac{1}{\sqrt{-g}}\,\partial_\rho\left(\sqrt{-g}\,g^{\rho\sigma}\partial_\sigma\right)A.$$

[1] *S. Deser, R.P. Woodard*, Phys. Rev. Lett. **99** (2007) 111301, `arXiv:0706.2151`

The nonlocal action (1) can be rewritten in the "localized" form by introducing two scalar fields η and ξ:

$$\tilde{S}_2 = \int d^4x \frac{\sqrt{-g}}{16\pi G_N} \left\{\left[R\left(1 + f(\eta)\right) + \xi\left(\Box\eta - R\right) - 2\Lambda\right] + \mathcal{L}_{\mathrm{m}}\right\}. \qquad (2)$$

By the variation over ξ, we obtain $\Box\eta = R$.
Substituting $\eta = \Box^{-1}R$ into (2), one reobtains action (1).

S. Nojiri, S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821, arXiv:0708.0924

We studied the de Sitter and power-low solutions of this model
E. Elizalde, E. O. Pozdeeva and S. Y. Vernov, Phys. Rev. D **85** (2012) 044002, [arXiv:1110.5806 [astro-ph.CO]].
E. Elizalde, E. O. Pozdeeva and S. Y. Vernov, Class. Quant. Grav. **30** (2013) 035002 [arXiv:1209.5957 [astro-ph.CO]].
E. Elizalde, E. O. Pozdeeva, S. Y. Vernov and Y. l. Zhang, JCAP **1307** (2013) 034 [arXiv:1302.4330 [hep-th]].

Non-local models with the Gauss–Bonnet term

Non-local models with the Gauss-Bonnet term have been proposed in
S. Capozziello, E. Elizalde, S. Nojiri, and S.D. Odintsov, *Phys. Lett.* B **671** (2009) 193, [arXiv:0809.1535],
where accelerating cosmological solutions have been studied.
Also, a localization procedure was proposed in this paper.
We continue to investigate this class of non-local models, and check for the existence of de Sitter and power-law solutions.

Non-local models with the Gauss–Bonnet term and their localization

We consider the non-local model with the Gauss–Bonnet term $\mathcal{G}$:

$$S_{NL} = \int dx^4 \sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{16\pi}R + C\mathcal{G}^{n_1}\Box^{-n_2}\mathcal{G}^{n_3} - \Lambda\right], \tag{3}$$

where M_{Pl} is the Planck mass, C and Λ are constants,
n_k are natural numbers, and the Gauss–Bonnet term

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.$$

$R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar.
$\Box$ is the d'Alembertian operator in the metric $g_{\mu\nu}$ acting on a scalar.
Using a localization procedure we can present action in the form

$$S_{L1} = \int dx^4 \sqrt{-g}\left[\frac{M_{Pl}^2}{16\pi}R + C\mathcal{G}^{n_1}\phi_{n_2} - \xi_1\mathcal{G}^{n_3} + \sum_{j=1}^{n_2}\phi_j\Box\xi_j - \sum_{j=1}^{n_2-1}\xi_{j+1}\phi_j - \Lambda\right] \tag{4}$$

Action S_L can be linearized with respect to the Gauss-Bonnet term, by adding one more scalar field in the action [2]). Let us consider the part of action S_L that includes the Gauss-Bonnet term:

$$S_{fGB} = \int dx^4 \sqrt{-g}\left[C\mathcal{G}^{n_1}\phi_{n_2} - \xi_1 \mathcal{G}^{n_3}\right]. \tag{5}$$

To linearize this action with respect to $\mathcal{G}$ we introduce a scalar field σ and

$$f(\sigma) = C\sigma^{n_1}\phi_{n_2} - \xi_1\sigma^{n_3}, \tag{6}$$

and get that the following equivalent action:

$$S_{GB\sigma} = \int dx^4 \sqrt{-g}\left[\frac{df}{d\sigma}(\mathcal{G}-\sigma) + f\right] =$$
$$= \int dx^4 \sqrt{-g}\left[\left(n_1 C\sigma^{n_1-1}\phi_{n_2} - n_3\xi_1\sigma^{n_3-1}\right)(\mathcal{G}-\sigma) + C\sigma^{n_1}\phi_{n_2} - \xi_1\sigma^{n_3}\right].$$

Varying over σ, one gets $\sigma = \mathcal{G}$ and the action S_{fGB}. Note that the scalar field σ is not dynamical, because it has no kinetic term.

[2] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D **73** (2006) 084007, [hep-th/0601008]

So, the initial action S_{NL} can be written in the following scalar-tensor form:

$$S = \int dx^4 \sqrt{-g} \left[\frac{M_{Pl}^2}{16\pi} R + F\mathcal{G} - V - \sum_{k=1}^{n_2} g^{\mu\nu} \partial_\mu \xi_k \partial_\nu \phi_k \right] \tag{7}$$

where we use the following redesignation

$$\begin{aligned} F &= n_1 C \sigma^{n_1 - 1} \phi_{n_2} - n_3 \xi_1 \sigma^{n_3 - 1} \,, & (8) \\ V &= -C\sigma^{n_1} \phi_{n_2}(1 - n_1) - \xi_1 \sigma^{n_3}(n_3 - 1) + \sum_{k=1}^{n_2 - 1} \xi_{i+1}\phi_i + \Lambda \,. & (9) \end{aligned}$$

Friedmann equations

We consider the spatially flat FLRW universe with the interval

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right). \tag{10}$$

In this metric one gets

$$R = 6(\dot{H} + 2H^2), \quad \mathcal{G} = 24H^2(\dot{H} + H^2), \quad \Box F = -3H\dot{F} - \ddot{F},$$

where the Hubble parameter $H = \dot{a}/a$ and dots mean the time derivatives.

Field and Friedmann equations

In the FLRW metric the field equations have the following form

$$\ddot{\phi}_k = -3H\dot{\phi}_k + \mathcal{G}\frac{\partial F}{\partial \xi_k} - \frac{\partial V}{\partial \xi_k}, \qquad \ddot{\xi}_k = -3H\dot{\xi}_k + \mathcal{G}\frac{\partial F}{\partial \phi_k} - \frac{\partial V}{\partial \phi_k}\,. \quad (11)$$

The Friedmann equations read as follows

$$3H^2\frac{M_{Pl}^2}{16\pi} - \frac{1}{2}\sum_{k=1}^{n_2}\left(\dot{\phi}_k\dot{\xi}_k\right) - \frac{1}{2}V = -12H^3\dot{F}, \quad (12)$$

$$-\left(3H^2 + 2\dot{H}\right)\frac{M_{Pl}^2}{16\pi} - 8H\left(H^2 + \dot{H}\right)\dot{F} - 4H^2\ddot{F} - \frac{1}{2}\sum_{k=1}^{n_2}\dot{\phi}_k\dot{\xi}_k + \frac{V}{2} = 0. \quad (13)$$

Subtracting (13) from (12), we get

$$8H^3\dot{F} - 4\Box F H^2 + \frac{3M_{Pl}^2}{8\pi}H^2 + 8H\dot{F}\dot{H} + \frac{M_{Pl}^2}{8\pi}\dot{H} - V = 0. \quad (14)$$

Note that Eqs. (13) and (14) are third order differential equations with respect to the Hubble parameter.

Search for de Sitter solutions

If the Hubble parameter is a constant: $H = H_0$, then the Gauss-Bonnet term reads $\mathcal{G} = 24H_0^4 \equiv \mathcal{G}_0$ and $\sigma = \mathcal{G}_0$. As a consequence, the corresponding field equations get transformed into the following system of linear first order differential equations, with constant coefficients,

$$\begin{aligned}
\dot{\phi}_1 &= \psi_1, \\
\dot{\psi}_1 &= -3H_0\psi_1 - \mathcal{G}_0^{n_3}, \\
\dot{\phi}_j &= \psi_j, \qquad j = 2, \ldots, n_2, \\
\dot{\psi}_j &= -3H_0\psi_j - \phi_{j-1}, \qquad j = 2, \ldots, n_2.
\end{aligned} \tag{15}$$

The system (15) has the following solution

$$\phi_j = P_j(t)e^{-3H_0 t} - \frac{\mathcal{G}_0^{n_3}}{j!(3H_0)^j}t^j + \tilde{P}_j(t), \tag{16}$$

where $P_j(t)$ and $\tilde{P}_j(t)$ are $(j-1)$-degree polynomials of t with coefficients that include $2j$ arbitrary parameters.

Analogously, the second system of field equations can be presented in the form

$$\begin{aligned} \ddot{\xi}_j + 3H_0\dot{\xi}_j + \xi_{j+1} &= 0, \qquad j = 1, \ldots, n_2 - 1, \\ \ddot{\xi}_{n_2} + 3H_0\dot{\xi}_{n_2} - C\mathcal{G}_0^{n_1} &= 0, \end{aligned} \tag{17}$$

and the solution reads

$$\xi_j = Q_j(t)e^{-3H_0 t} + \frac{C\mathcal{G}_0^{n_1}}{(n_2 - j + 1)!(3H_0)^{n_2 - j + 1}} t^{(n_2 - j + 1)} + \tilde{Q}_j(t), \tag{18}$$

where $Q_j(t)$ and $\tilde{Q}_j(t)$ are polynomials in t of degree $(n_2 - j) + 1$.
To check for the existence of de Sitter solutions, one must substitute the solutions of the field equations thus obtained into Eqs. (12) and (14).

In the cases when $n_1 = n_2 = n_3 = 1$ and $n_1 = -n_2 = n_3 = -1$ the Sitter solutions have been found in [3]. To get de Sitter solution in the model is non-trivial problem. For example, we checked that de Sitter solution is absent in case $n_1 = 1$, $n_2 = 2$, $n_3 = 1$. We obtain de Sitter solution the case $n_2 = 2$.

[3] S. Capozziello, E. Elizalde, S. Nojiri, and S.D. Odintsov, *Phys. Lett.* B **671** (2009) 193, [arXiv:0809.1535]

case $n_2 = 2$

Let us consider the case $n_2 = 2$.
The field equations

$$-\ddot{\phi}_1 - 3H_0\dot{\phi}_1 = \mathcal{G}_0^{n_3}, \qquad -\ddot{\phi}_2 - 3H_0\dot{\phi}_2 = \phi_1 \tag{19}$$

and

$$-\ddot{\xi}_2 - 3H_0\dot{\xi}_2 = -C\mathcal{G}_0^{n_1}, \qquad -\ddot{\xi}_1 - 3H_0\dot{\xi}_1 = \xi_2 \tag{20}$$

have the following solutions:

$$\phi_1 = A_1 e^{-3H_0 t} - \frac{\mathcal{G}_0^{n_3}}{3H_0} t + B_1, \tag{21}$$

$$\phi_2 = \left(\frac{A_1}{3H_0} t + A_2\right) e^{-3H_0 t} + \frac{\mathcal{G}_0^{n_3}}{18H_0^2} t^2 - \left(\frac{\mathcal{G}_0^{n_3}}{27H_0^3} + \frac{B_1}{3H_0}\right) t + B_2 \tag{22}$$

$$\xi_1 = \left(\frac{C_1}{3H_0} t + C_2\right) e^{-3H_0 t} - \frac{C\mathcal{G}_0^{n_1}}{18H_0^2} t^2 + C\left(\frac{\mathcal{G}_0^{n_1}}{27H_0^3} - \frac{D_1}{3H_0}\right) t + D_2, \tag{23}$$

$$\xi_2 = C_1 e^{-3H_0 t} + C\frac{\mathcal{G}_0^{n_1}}{3H_0} t + CD_1, \tag{24}$$

where A_i, B_i, C_i, and D_i are integration constants.

Substituting these expressions into Eq. (12),

$$3H_0^2\frac{M_{Pl}^2}{16\pi}-\frac{1}{2}\left(\sum_{k=1}^{n_2}\dot{\phi}_k\dot{\xi}_k\right)-\frac{1}{2}V=-12H_0^3\dot{F}, \tag{25}$$

we see that this equation can be satisfied only if $n_1+n_3=4$. Also, we get the following restriction to the integration constants

$$A_1=0,\quad C_1=0,\quad C_2=-\frac{24^{2n_1}(2n_1-1)A_2CH_0^{8(n_1-2)}}{331776(2n_1-7)}, \tag{26}$$

$$B_1=-\frac{331776(n_1-2)H_0^{16-8n_1}D_1+24^{n_1}442368H_0^{14-4n_1}}{(n_1-2)}. \tag{27}$$

These restrictions are not valid for $n_1=2$.

Also, we have the additional to connect the values of the parameters of the solutions, with Λ

$$\begin{aligned}\Lambda=&-\frac{3H_0^2M_{Pl}^2}{8\pi}-\frac{8192C(13n_1+4)H_0^{12}}{(n_1-2)}\\&+24^{-n_1}331776D_2(n_1-3)H_0^{16-4n_1}+24^{n_1}CB_2(n_1-1)H_0^{4n_1}\\&-\frac{24^{-n_1}73728(5n_1-4)CD_1H_0^{14-4n_1}}{n_1-2}-24^{-2n_1}331776CD_1^2H_0^{16-8n_1}.\end{aligned} \tag{28}$$

Consequently, the value of Λ fixes the value of one of the integration constants: B_2 for $n_1 = 3$ or D_2 for $n_1 = 1$.
Summing up, we do get explicitly de Sitter solutions for models with $n_1 = 1$, $n_2 = 2$, $n_3 = 3$ and $n_1 = 3$, $n_2 = 2$, $n_3 = 1$.
We have also discovered that models with $n_2 = 2$ and other values of n_1 and n_3 do not have de Sitter solutions.
Straightforward substitution of the field expressions when $n_1 = n_3 = 2$ already proves the absence of the de Sitter solutions in this case.

Power Law solutions

The search of power-law solutions with $H = J/t$ is more complicated. We consider the case when n_1 or n_3 is equal to 1. If $n_1 = 1$ and $n_3 = 1$, then

$$V = \xi_2 \phi_1, \qquad F = C\phi_2 - \xi_1 \tag{29}$$

with the following form for the field equations

$$\Box\phi_1 = \mathcal{G}, \qquad \Box\phi_2 = \phi_1, \tag{30}$$

$$\Box\xi_2 = -C\mathcal{G}, \qquad \Box\xi_1 = \xi_2, \tag{31}$$

where $\mathcal{G} = 24(J-1)J^3/t^4$.
Using these formulas, we immediately obtain the form of Eq. (14)

$$-2\left(3H^2 + \dot{H}\right)\frac{M_{Pl}^2}{16\pi} - 8H\left(H^2 + \dot{H}\right)(C\dot{\phi}_2 - \dot{\xi}_1) + 4(C\phi_1 - \xi_2)H^2 + \xi_2\phi_1 = 0 \tag{32}$$

The model with $n_1 = 1$ and $n_3 = 1$ yields power law solutions with $H = J/t$ at $J = 2/3$ and $J = 3$. The corresponding scalar fields admit two types of expressions.

The first type of solutions corresponds to

$$\begin{aligned}
\phi_1 &= -\frac{C_1\, t^{-3H_0+1}}{3J-1} + 4\,\frac{J^3}{t^2} - \frac{1}{2}\,\frac{K\,(3J+1)}{J\,C\,(J-1)} \\
\phi_2 &= \frac{1}{4}\,\frac{t^2 K}{J\,C\,(J-1)} - \frac{K_3\, t^{-3J+1}}{C\,(3J-1)} - \frac{1}{6}\,\frac{C_1\, t^{3-3J}}{3\,J^2-4\,J+1} - 4\,\frac{J^3 \ln(t)}{3J-1} + C_4 \\
\xi_1 &= 4\,\frac{CJ^3 \ln(t)}{3J-1} - \frac{K_3\, t^{-3J+1}}{3J-1} + K_4 \\
\xi_2 &= -4\,\frac{CJ^3}{t^2},
\end{aligned}$$

where in the case $J = 2/3$, $C_1 = \frac{7168}{729\, C_3}$, while in the case $J = 3$, either $C_1 = 0$ or $C_3 = 0$.

Another type of solutions, with the same Hubble parameters, is given by

$$
\begin{aligned}
\phi_1 &= 4\,\frac{J^3}{t^2}, \\
\phi_2 &= -\frac{C_3\,t^{-3J+1}}{3J-1} - 4\,\frac{J^3 \ln(t)}{3J-1} + C_4, \\
\xi_1 &= 4\,\frac{CJ^3 \ln(t)}{3J-1} - \frac{C\,C_3\,t^{-3J+1}}{3J-1} - \frac{1}{4}\,\frac{t^2 K}{J\,(J-1)} - \frac{1}{6}\,\frac{t^{3-3J} K_1}{3\,J^2 - 4\,J + 1} + K_4, \\
\xi_2 &= -\frac{K_1\,t^{-3J+1}}{3J-1} - 4\,\frac{CJ^3}{t^2} + \frac{1}{2}\,\frac{K\,(3J+1)}{J\,(J-1)},
\end{aligned}
$$

where in the case $J = 2/3$ we have the additional condition $K_1 = -\frac{7168\,C}{729\,C_3}$, while in the case $J = 3$, either $C_3 = 0$ or $K_1 = 0$. Note that the form of the solutions obtained excludes a few values of J, which must be checked separately.

Conclusions

We analyze the Gauss-Bonnet non-local gravity model:

$$S_{NL} = \int dx^4 \sqrt{-g} \left[\frac{M_{Pl}^2}{16\pi} R + C\mathcal{G}^{n_1} \Box^{-n_2} \mathcal{G}^{n_3} + \mathcal{L}_m \right].$$

and obtain

- in the specific case $n_2 = 2$, de Sitter solutions exist only in these two cases: for $n_1 = 1$ and $n_3 = 3$, or for $n_1 = 3$ and $n_3 = 1$. Both these models yield no power-law solutions;
- if $n_1 = 1$ and $n_3 > 1$ (or $n_1 > 1$ and $n_3 = 1$, respectively), then power-law solutions do not exist;
- in the case $n_1 = n_3 = 1$, power-law solutions with $H = J/t$ exist only for $J = 2/3$ and $J = 3$. Therefore, the model with $n_1 = 1$, $n_2 = 2$, and $n_3 = 1$, without additional matter, is suitable in order to describe the matter-dominated phase of the Universe evolution that corresponds to $J = 2/3$.

Thank for your attention