Stabilization Of Extra Dimensions In Nonlinear Multidimensional Gravity With Multiple Factor Spaces

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Motivation

- The idea of extra dimensions is a powerful methodological framework in modern theoretical physics (Kaluza–Klein theories, superstrings, brane worlds, etc.).
- There are many variants of multidimensional gravity models including $F(R)$-like theories and others.
- Among them, it is interesting to search for realistic models explaining the unobservable nature of extra dimensions, possible mechanisms of their stabilization, and providing the agreement with the realistic cosmological scenarios and observations.

The main objectives

- Analysis of the properties of an effective potential in a Kaluza-Klein-type model with curvature-nonlinear terms and multiple extra factor spaces.
- Demonstration of the existence of physically appropriate minima of the potential, providing stabilization of compact extra dimensions in agreement with the observed accelerated expansion of the Universe.

The model

Consider a Kaluza-Klein-type model with curvature-nonlinear terms and a number of extra spaces $\mathbb{M}_i$ $(i = \overline{1, N})$ with constant nonzero curvature of sign $K_i = \pm 1$ and a fixed curvature radius r_0, normalized to the D-dimensional analogue m_D of the Planck mass.

Manifold: $\mathbb{M} = \mathbb{M}_0 \times \mathbb{M}_1 \times \cdots \times \mathbb{M}_N$, $\dim \mathbb{M}_i = d_i \quad (d_0 = 4)$

Action:

$$S_D = \frac{m_D^{D-2}}{2} \int \sqrt{g_D}\, d^D x \Big[F(R) + c_1 R^{AB} R_{AB} + c_2 R^{ABCD} R_{ABCD} + L_m\Big]$$

The metric: $$ds_D^2 = g_{ab}(x) dx^a dx^b + \sum_{i=1}^{N} \mathrm{e}^{2\beta_i(x)} g^{(i)}$$

$x \equiv (x^a)$ are coordinates in $\mathbb{M}_0$; $g_{ab} = g_{ab}(x)$ is the metric in $\mathbb{M}_0$;
$g^{(i)}$ are x-independent d_i-dimensional metrics of factor spaces $\mathbb{M}_i$;
$m_D \equiv 1/r_0, \quad g_D = |\det(g_{MN})|$

Methodology

(a) Dimensional reduction in the Jordan frame: $(4+d)$-splitting of all curvature-related quantities and integration of the D-dimensional action over compact extra space to obtain the 4D action containing effective potential of scalar fields $\beta_i(x^\mu)$ which determine the size of extra dimensions.

(b) Slow change approximation: we suppose that all quantities are slowly varying as compared with the D-dimensional Planck scale, i.e., consider each derivative ∂_μ as an expression with a small parameter ε and neglect all quantities of orders higher than $O(\varepsilon^2)$.

(c) Transition to the Einstein frame where the fields $\phi_i \sim \exp(-2\beta_i)$ are minimally coupled to the 4D curvature.

(d) Estimation of possible Casimir contribution of scalar fields from some numerical results and dimensional analysis considerations.

(e) Search for stable minima of the effective potential of scalar fields β_i under some realistic physical assumptions

Reduced action:

$$S_J = \frac{1}{2}\mathcal{V}\, m_D^{d_0-2} \int \sqrt{g_0}\, d^{d_0}x \left\{ \mathrm{e}^{\sigma} F'(\phi) R_0 + K_J \; - 2\left[V_J(\phi_i) + V_{J(\mathrm{Cas})}\right] \right\}$$

Kinetic term:

$$K_J = F'\,\mathrm{e}^{\sigma}\left[-(\partial\sigma)^2 + \sum_i d_i (\partial\beta_i)^2 - 2F''(\partial\phi,\partial\sigma)\right] + 4\,\mathrm{e}^{\sigma}(c_1+c_2)\sum_i d_i\phi_i(\partial\beta_i)^2$$

Effective potential:

$$-2V_J(\phi_i) = \mathrm{e}^{\sigma}\left[F(\phi) + \sum_i d_i\phi_i^2\left(c_1 + \frac{2c_2}{d_i-1}\right)\right]$$

$g_0 = |\det(g_{ab})|, \quad \Box = g^{ab}\nabla_a\nabla_b$
$\phi_i := K_i m_D^2 (d_i-1)\,\mathrm{e}^{-2\beta_i}, \quad \phi := \sum_i d_i\phi_i$
$\sigma := \sum_i d_i\beta_i, \quad (\partial\sigma)^2 \equiv \sigma_{,a}\sigma^{,a}, \quad (\partial\alpha,\partial\beta) \equiv g^{ab}\alpha_{,a}\beta_{,b}$
$\mathcal{V}$ is a product of volumes of N compact spaces $\mathbb{M}_i$ of unit curvature

Dimensional reduction: The Einstein frame

Use the CONFORMAL MAPPING: $g_{ab} \mapsto \widetilde{g}_{ab} = |\,e^{\sigma} F'(\phi)|^{2/(d_0-2)} g_{ab}$

Reduced action:

$$S_E = \frac{1}{2}\mathcal{V}\, m_D^{d_0-2} \int \sqrt{\widetilde{g}}\, d^{d_0}x \Big\{ [\operatorname{sign} F'(\phi)] \Big[\widetilde{R} + K_E\Big] - 2[V_E(\phi_i) + V_{E(\mathrm{Cas})}] \Big\}$$

Kinetic term:

$$K_E = \frac{1}{d_0-2}\left(\partial\sigma + \frac{F''}{F'}\partial\phi\right)^2 + \left(\frac{F''}{F'}\right)^2 (\partial\phi)^2 + \sum_i d_i \left[1 + \frac{4}{F'}(c_1+c_2)\phi_i\right](\partial\beta_i)^2$$

Effective potential:

$$-2V_E(\phi_i) = e^{-2\sigma/(d_0-2)} |F'|^{-d_0/(d_0-2)} \left[F(\phi) + \sum_i d_i \phi_i^2 \left(c_1 + \frac{2c_2}{d_i-1}\right)\right]$$

Physical assumptions

- We describe space-time classically, therefore the size of the extra dimensions $r_i = r_0 \, \mathrm{e}^{\beta_i}$ should appreciably exceed the fundamental length scale $r_0 = 1/m_D$, i.e., $\boxed{r_i/r_0 = \mathrm{e}^{\beta_i} \gg 1}$.
- The extra dimensions should not be directly observable, which means that $\boxed{r_i = r_0 \, \mathrm{e}^{\beta_i} \lesssim 10^{-17}\,\mathrm{cm}}$, which is close to the TeV energy scale.
- The effective 4D cosmological constant $\Lambda_{\rm eff}$ corresponding to the minimum value of the effective potential should conform to observational constraints: $\boxed{0 < \Lambda_{\rm eff}/m_4^2 \sim 10^{-120}}$, where $m_4 \sim 10^{-5}\,\mathrm{g}$ is the 4D Planck mass.
- Non-phantom character of the scalar fields, which means that the kinetic term should be positive definite: $\boxed{K_E(\phi_1, ..., \phi_N) > 0}$.
- Here we restrict our analysis to the case of two extra factor spaces $\mathbb{M}_1 = \mathbb{S}^n$ and $\mathbb{M}_2 = \mathbb{S}^m$.

Effective potential and kinetic term on $\mathbb{M}^4 \times \mathbb{S}^n \times \mathbb{S}^m$

- The function $F(R) = -2\Lambda_D + R$
- **Dimensionless potential** $W(x,y) \equiv r_0^2 V_{\rm E}(x,y)$:

$$W(x,y) = \frac{x^m y^n}{2}\Big[k_1 x^4 + k_2 y^4 - m(m-1)x^2 - n(n-1)y^2 + \lambda\Big] + W_{\rm Cas}$$

- **Kinetic term**:

$$K(x,y) = \frac{m}{2}(\partial x)^2\left[\frac{m+2}{x^2} + 8(m-1)(C_1+C_2)\right] + \frac{n}{2}(\partial y)^2\left[\frac{n+2}{y^2} + 8(n-1)\,(C_1+C_2)\right] + \frac{mn(\partial x,\partial y)}{xy}$$

where the following notations are used:

$x = {\rm e}^{-\beta_1}, \quad y = {\rm e}^{-\beta_2}, \quad C_1 = c_1/r_0^2, \quad C_2 = c_2/r_0^2, \quad \lambda = r_0^2\Lambda_D,$

$k_1 = -\frac{m(m-1)}{2}\big[C_1(m-1)+2C_2\big], \quad k_2 = -\frac{n(n-1)}{2}\big[C_1(n-1)+2C_2\big]$

Approximate estimate of Casimir energy

For estimation purposes we can use the expression for $V_{J(\mathrm{Cas})}$ on $\mathbb{M}_0 \times \mathbb{S}^3 \times \mathbb{S}^3$ found for the case of two spheres $\mathbb{S}^3$ of approximately equal radii [Gleiser et al., 1987], $r_2/r_1 = 1+\epsilon$, where $\epsilon \lesssim 0.25$, and, in our notations, $r_1 = r_0 e^{\beta_1}$, $r_2 = r_0 e^{\beta_2}$:

$$
\begin{aligned}
V^{(3,3)}_{J(\mathrm{Cas})} &= \\
&= -\frac{1}{r_1^4 \mathcal{V}} \Big[b\Big(3.639\times 10^{-4} - 6.053\times 10^{-4}\epsilon \\
&+ 3.315\times 10^{-4}(1-2\epsilon)\ln(r_1/\bar{\mu})\Big) \\
&+ f(3.657\times 10^{-6} - 5.414\times 10^{-5}\epsilon)\Big],
\end{aligned}
$$

where b and f are the numbers of spin-0 and spin-1/2 fields, and $\bar{\mu}$ is parameter emerging in the renormalization procedure.

From other (mostly numerical) calculations [Kikkawa et al., 1985; Gleiser et al., 1987; Birmingham et al., 1988], it follows that $V_{J(\mathrm{Cas})}$ can have approximately the order of magnitude $\sim 10^{-4} \times r_1^{-4}$ times the number of field degrees of freedom. A rough estimation yields (restoring the symmetry between $r_1 = r_0 x^{-1}$ and $r_2 = r_0 y^{-1}$):

$$V_{J(\mathrm{Cas})} \lesssim \mathrm{const} \cdot r_0^{-4} x^2 y^2$$

$$W_{\mathrm{Cas}}(x,y) \lesssim \mathrm{const} \cdot x^{2m+2} y^{2n+2}$$

Such a contribution is insignificant for the presence of a local minimum of W in the semiclassical region ($x \ll 1, y \ll 1$), but can affect the value of $W(x_0, y_0)$ related to the Λ_{eff}.

The stable equilibrium

Consider manifold $\mathbb{M}^4 \times \mathbb{S}^n \times \mathbb{S}^m$ with equal dimensions of extra factor spaces: $\boxed{m = n}$.

It is not difficult to demonstrate the existence for small stable positive minima of $W(x, y)$ in the region $\boxed{x_{\min} \lesssim 0.1,\ y_{\min} \lesssim 0.1}$ choosing appropriate values of constants c_1, c_2, λ and fulfilling the requirement for the kinetic term to be a *positive definite*.

Assertion

For all $m = n \neq 3$ there exists a local stable minimum of the potential $W(x_0, y_0)$ at proper "semiclassical" values of $x_0 = y_0 \sim 0.01$.
The kinetic term can be shown to be positive-definite under the condition $c_1 + c_2 \geq 0$.
The critical value of λ that $W(x_0) = 0$ is:

$$\lambda_c = -\frac{n(n-1)}{4C_1(n-1) + 8C_2}$$

The function $W(x, y)$ close to its minimum for $m = n = 5$,
$C_1 = -127.25$, $C_2 = 128$, $\lambda = 0.0197628$.
At $x = y$ we have $W(x, x) = x^{10}(0.0197628 - 20x^2 + 5060x^4)$.

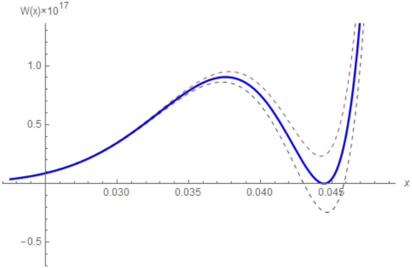

Figure 1: $W(x, x)$ for the parameters indicated above (solid line), and a bit larger and smaller λ (dashed lines)

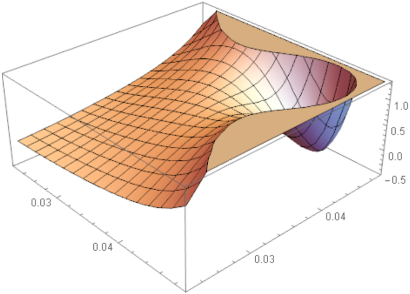

Figure 2: The 3D plot of the $W(x, y)$ for the parameters indicated above

Some estimations

- **The Einstein frame:** The 4D Planck mass is $m_4 = \sqrt{\mathcal{V}(n)}m_D$, where $m_D = r_0^{-1}$. As long as $\mathcal{V}(n)$ is not very far from unity for a wide range of n, $r_0 = \sqrt{\mathcal{V}}m_4^{-1} \sim 10^{-31}\mathrm{cm}$, and for $x_0 = y_0 \sim 0.01$ the size of extra dimensions is $\boxed{r(x_0) = r_0 x_0^{-1} \sim 10^{-29}\mathrm{cm}}$.
- **The Jordan frame:** $m_D^2 = m_4^2 x_0^{2n} \mathcal{V}^{-1} \Longrightarrow r_0 = \sqrt{\mathcal{V}} m_4^{-1} x_0^{-n}$. As $x_0 \ll 1$, the size of extra of both extra spaces $r_1 = r_2 = r_0 x_0^{-1}$ may be in tension with the invisibility of extra dimensions at large enough $n > 8$.
  For $n = 8:$ $\boxed{r_0 \approx 1.1\times 10^{-19}\,\mathrm{cm},\ \ r_1 \approx 3\times 10^{-18}\,\mathrm{cm}.}$
- **Cosmological constant**: $\Lambda_4^{\mathrm{eff}} = W(x_0) r_0^{-2} = W(x_0) m_4^2 \mathcal{V}^{-1}$. To conform the observational requirement $\Lambda_4/m_4^2 \sim 10^{-120}$, the fine tuning is necessary: the parameter $\lambda = r_0^2 \Lambda_D$ should be close to the value at which $W(x_0) = 0$.
  The Casimir contribution to $W(x_0)$ is large as compared to 10^{-120} (for instance, $W_{\mathrm{Cas}} \sim 10^{-24}$ for $n = 5$), and can be compensated by fine-tuned values of other parameters of the theory, above all, λ.

Conclusion

- Within the scope of multidimensional Kaluza–Klein gravity with nonlinear curvature terms and two spherical extra spaces $\mathbb{S}^m$ and $\mathbb{S}^n$, we studied the properties of an effective action for the scale factors of the extra dimensions.
- Dimensional reduction leads to an effective 4D multiscalar-tensor theory in the Jordan and the Einstein conformal frames.
- Based on qualitative estimates of the Casimir energy contribution on a physically reasonable length scale, we demonstrated the existence of such sets of initial parameters of the theory in the case $m = n$ that provide a minimum of the effective potential that yields a fine-tuned value of the effective 4D cosmological constant.
- The corresponding size of extra dimensions depends of which conformal frame is interpreted as the observational one: it is about three orders of magnitude larger than the standard Planck length if we adhere to the Einstein frame, but it is n-dependent in the Jordan frame, and its invisibility requirement restricts the total dimension $D = 4 + 2n \lesssim 20$.

References

- K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero Point Energy* (World Scientific, Singapore, 2001)
- P. Candelas and S. Weinberg, Nucl. Phys. B **237**, 397 (1984).
- S. V. Bolokhov and K. A. Bronnikov, "On nonlinear multidimensional gravity and the Casimir effect", Grav. Cosmol. **22**, 323 (2016).
- Keiji Kikkawa, Takahiro Kubota, Shiro Sawada and Masami Yamasaki, Nucl. Phys. **B** 260, 429 (1985)
- M. Gleiser, Ph. Jetzer, and Mark A. Rubin, Phys. Rev. D **36**, 2429 (1987)
- D. Birmingham, R. Kantowski, and K. A. Milton, Phys. Rev. D **38**, 1809 (1988)
- C. Wetterich, Phys. Lett. B **113**, 377 (1982).
- K.A. Bronnikov, S.G. Rubin, and I.V. Svadkovsky, Phys. Rev. D **81**, 084010 (2010); ArXiv: 0912.4862.

Thank you for your attention!