Operator-based method for finding exact solutions of the propagator equation in the presence of a magnetic field

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Abstract

Elementary particle processes in the extreme astrophysical conditions, such as strong magnetic fields, require knowledge of the exact propagators. There are known expressions for the propagators of scalar, Dirac and massive vector fields in the presence of a constant magnetic field both in the coordinate and in the momentum spaces. In general they require either following the tedious Fock-Schwinger procedure or first obtaining the exact solutions of the wave equation of interest followed by summation over the allowed quantum numbers. In this work we present a general method of obtaining the exact analytical solutions of the propagator equation based on the decomposition of the delta function into the sum of the Hamiltonian-like operator eigenfunctions with the subsequent integration of the corresponding operator exponent in the proper time domain. Providing that parts of the operator exponent commute, it becomes possible to decouple them from each other and apply each part separately to the delta function decomposition series. This method not just allows to straightforwardly obtain the expression for the propagator in the momentum space as a sum over the Landau levels, but also helps to gain insights into the propagator's anatomy, revealing the origins of its constituent parts.

Outline of the Fock-Schwinger approach

To obtain the expression of the propagator satisfying equation

$$H(\partial_x, x)G(x,x') = \delta^4(x-x') \quad (1)$$

using the Fock-Schwinger method (e.g., see [1]) one should stick to the following steps. First, the propagator *G(x, x')* is represented as an integral:

$$G(x,x') = (-i)\int_{-\infty}^{0} d\tau\, U(x,x';\tau) \quad (2)$$

where τ is called the *proper time*. Considering *U(x,x';τ)* as some sort of an evolution operator satisfying a Schrödinger-type equation

$$i\partial_t U(x,x';\tau) = H(\partial_x,x)U(x,x';\tau) \quad (3)$$

with the appropriate boundary conditions

$$U(x,x';-\infty) = 0 \qquad U(x,x';0) = \delta^4(x-x') \quad (4)$$

one obtains the following expression:

$$U(x,x';\tau) = exp\big(-i\tau H(\partial_x,x)\big)\,\delta^4(x-x') \quad (5)$$

Next, we represent (3) using (5) and the common notation for the bra-ket product in the coordinate space:

$$\langle x|A|x'\rangle = \int d^4X\ \delta^4(X-x)\ A(X)\ \delta^4(X-x') \quad (6)$$

From that we get the following equation:

$$i\,\partial_t\langle x|e^{-i\tau H}|x'\rangle = \langle x|He^{-i\tau H}|x'\rangle = \langle x|e^{-i\tau H}e^{+i\tau H}He^{-i\tau H}|x'\rangle \quad (7)$$

In some special cases it is possible to factor out a scalar function *F(x,x'; τ)*:

$$i\partial_t\langle x|e^{-i\tau H}|x'\rangle = F(x,x';\tau)\langle x|e^{-i\tau H}|x'\rangle \quad (8)$$

therefore making the equation easy to integrate:

$$U(x,x';\tau) = exp\big(-i\int d\tau\, F(x,x';\tau)\big)\,C(x,x') \quad (9)$$

In general this factorization is possible if one was able to make use of the commutation relations in the Heisenberg picture. It is worth noting that for the problem of charged particles in the presence of a constant electromagnetic field it is definitely the case. The final expression for *G(x, x')* will be a function in coordinate space but often it is needed to consider propagators in the momentum space as a Fourier decomposition. At least two possible scenarios were found so far. First, we could Fourier-transform the expression *G(x, x')*, which itself is a challenging task. To learn about existing techniques one could refer to [2] where the cases of scalar, massive vector and fermion fields were considered.

Another possibility, more straightforward and with clear physical meaning, is to construct the propagator from the exact solutions of the corresponding wave-equation. Being computationally trivial in the scalar case this task becomes more time consuming when considering vector and fermion fields which have additional degrees of freedom. Complexity arises when one needs to evaluate wave-function normalization, find the orthogonal set of solutions and to sum over the possible polarization states to obtain density matrix. If the wave-function is not itself of a great interest, but rather the propagator, it is possible to skip those steps by going directly to the expression with no sign of individual polarization states left.

Operator-based method

Looking at the expression (5) it becomes clear that the exchange of the operator-valued expression to the c-number function in the right-hand side is possible if one achieves to decompose δ-function as a series/integral of the eigenstates of the operator $H(\partial_x,x)$, hence the name of the approach. Replacing *H* in the exponent with its eigenvalue will result in the series/integral of the same structure but with different coefficients (which are, obviously, eigenvalues of *H* for each eigenstate in decomposition).

Let's look at the baby example, i.e. free scalar field:

$$(-\partial^2 - m^2)G(x,x') = \delta^4(x-x') \quad (10)$$

Normally it is solved considering the translational symmetry of the problem, i.e.:

$$G(x,x') = G(x-x') \quad (11)$$

with the subsequent Fourier decomposition of both *G* and the δ-function. But actually this knowledge is not required if we apply the proposed approach with just decomposing the δ-function and applying the operator exponent:

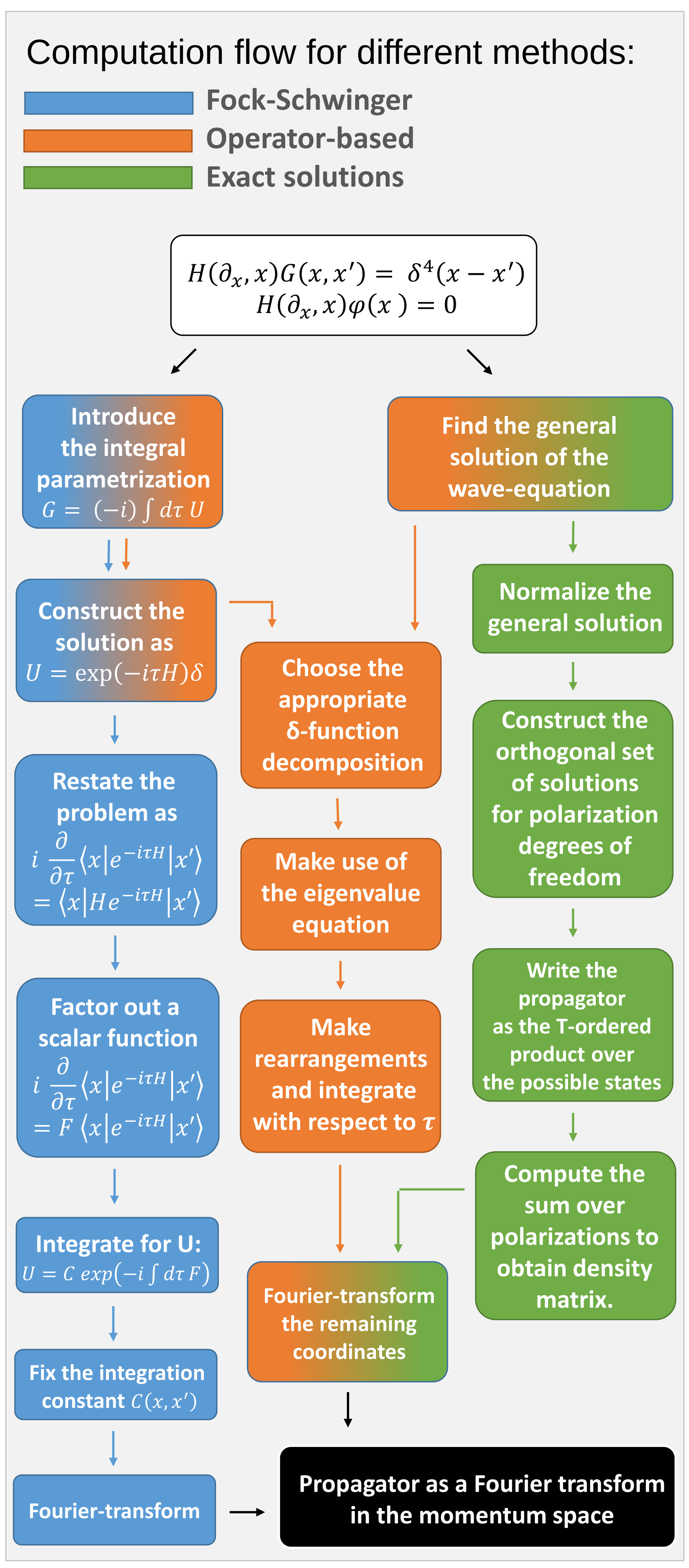
$$G(x,x') = (-i)\int_{-\infty}^{0} d\tau\, e^{-i\tau(-\partial^2-m^2+i\varepsilon)}\int\frac{d^4p}{(2\pi)^4}e^{-i(p(x-x'))} \quad (12)$$

From (12) it is clear that substitution $-\partial^2 \to p^2$ is possible. Finally, integration with respect to τ yields the well-known expression:

$$G(x,x') = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-i(p(x-x'))}}{p^2-m^2+i\varepsilon} \quad (13)$$

Here, we've followed the usual prescription adding *+iε* in the exponent, hence making the integral converge.

From this example the general idea of the operator-based method could be seen: it is only required to know which form the solution φ(x) of wave-equation has, in the above case $\varphi(x) \sim \exp(-ipx)$.



Electron in a magnetic field

The real utility of the method could be seen if we consider a somewhat harder problem, e.g. a charged fermion in the presence of a constant magnetic field. Choosing the appropriate gauge $A^\mu = (0,0,Bx,0)$ one could write the equation for the propagator of a charged fermion:

$$\big[(i\partial_\mu - eQA_\mu)\gamma^\mu - m\big]G(x,x') = I\,\delta^4(x-x') \quad (14)$$

One of the standard procedures of manipulating Dirac equation is to make it a second order equation. We apply the same trick here by representing *G* as:

$$G(x,x') = [(i\partial_\nu - eQA_\nu)\gamma^\nu + m]S(x,x') \quad (15)$$

From this one using the properties of gamma matrices we arrive at:

$$H(\partial_x,x)S(x,x') = I\,\delta^4(x-x') \quad (16)$$

with the *H* operator having the form:

$$H(\partial_x,x) = \Big[\big(p_0^2 - p_z^2 - m^2 + \beta(d_\xi^2 - \xi^2)\big)I + Q\beta\Sigma_3\Big] \quad (17)$$

Here the standard notations were used:

$$\beta = eB \qquad \xi = \sqrt{\beta}\left(x - Q\frac{p_y}{\beta}\right) \qquad \Sigma_3 = diag(+1,-1,+1,-1) \quad (18)$$

The following substitutions justified by the appropriate δ-function decomposition (21) were also made:

$$i\partial_0 \to p_0 \qquad -i\partial_y \to p_y \qquad -i\partial_z \to p_z \quad (19)$$

We see that (17) is nothing but an operator, describing a set of four harmonic oscillators with energies satisfying the relation:

$$p_0^2 = p_z^2 + m^2 + \beta(2n+1) \pm Q\beta \quad (20)$$

Having the information about the form of the solution of the wave-equation as the input to the operator-based method, we therefore can choose the decomposition of the δ-function as:

$$\delta^4(x-x') = \sqrt{\beta}\sum_{n=0}^{\infty}\int\frac{d^2p_\parallel dp_y}{(2\pi)^3}e^{-i(p(x-x'))_{\parallel,y}}V_n(\xi)V_n(\xi') \quad (21)$$

where ‖ stands for t and z components, and $V_n(\xi)$ are the harmonic oscillator eigenstates:

$$V_n(\xi) = (2^n\, n!\,\sqrt{\pi})^{-1/2}\, exp(-\xi^2/2)\, H_n(\xi) \quad (22)$$

Here, H_n are Hermite polynomials. The propagator reads then:

$$G(x,x') = (-i)\sqrt{\beta}\,[(i\partial_\nu - eQA_\nu)\gamma^\nu + m]\sum_{n=0}^{\infty}\int\frac{d^2p_\parallel dp_y}{(2\pi)^3}\int_{-\infty}^{0}d\tau$$
$$e^{-i\tau\left[\left(p_\parallel^2 - m^2 + i\varepsilon - \beta(2n+1)\right)I + Q\beta\Sigma_3\right]}\,e^{-i(p(x-x'))_{\parallel,y}}\,V_n(\xi)V_n(\xi') \quad (23)$$

In the exponent the eigenvalue relation was used:

$$\big(d_\xi^2 - \xi^2\big)\,V_n(\xi) = -(2n+1)V_n(\xi) \quad (24)$$

We see that, first, the identity matrix I commutes with Σ_3. Therefore we could split $e^{-i\tau[\dots]}$ into two exponents and evaluate them separately. Secondly, in order to have the same expression in the exponent for all values of the propagator matrix, we should shift the summation for some elements.
Finally, integrating out the exponent (assuming Q = -1) we get:

$$G(x,x') = \sqrt{\beta}\,[(i\partial_\nu + eA_\nu)\gamma^\nu + m]\sum_{n=0}^{\infty}\int\frac{d^2p_\parallel dp_y}{(2\pi)^3}\,\frac{e^{-i(p(x-x'))_{\parallel,y}}}{p_\parallel^2 - m^2 - 2\beta n + i\varepsilon}V \quad (25)$$

The matrix V here is $\mathbf{diag}\big(V_{n-1}(\xi)V_{n-1}(\xi'),\ V_n(\xi)V_n(\xi'),\ V_{n-1}(\xi)V_{n-1}(\xi'),\ V_n(\xi)V_n(\xi')\big)$.

The rest of calculations consist in applying the $[(i\partial_\nu + eA_\nu)\gamma^\nu + m]$ operator, carrying out the integration over dp_y and computing the inverse Fourier transform. These steps are the same as if we were constructing the propagator from the exact solutions and are not provided here. One could learn more about the calculation techniques from [2]. The main point here is that by using the operator-based method we skipped the orthogonalization and normalization steps along with summation over polarizations and went straight to the correct expression of the density matrix.

Anatomy of the propagator

Let's summarize the calculation recipe, pointing out at the parts which determine the propagator's anatomy:

1. Write the propagator in the following representation:
   $G(x,x') = (-i)\int_{-\infty}^{0} d\tau \exp[-i\tau H(\partial_x,x)]\,\delta^4(x-x')$
2. Find the form of the operator *H* eigenstates and use the appropriate δ-function decomposition so that its individual terms are the eigenstates of *H*. *It is actually the δ-function decomposition that defines the form of the propagator.*
3. Make use of the eigenvalue equation to replace terms in *H*. *This will give the appropriate coefficients to each of the terms in the δ-function decomposition.*
4. If possible, split the exponent into several parts that could be evaluated separately. *This will reveal the spin dependence of the propagator.*
5. If needed, shift the decomposition summation index for some parts of the propagator so that they share the same $\exp[-i\tau(\dots)]$ factor. *This will result in mixing of different eigenstates.*
6. Carry out integration with respect to τ. *That's how we get the denominator. Don't forget to include* $+i\varepsilon$ *for convergence.*
7. Having eliminated operators in the representation of the propagator, proceed with further simplifications depending on the actual problem.

Conclusion

This method is by no means a conceptually new way to obtain propagators. It is better to be considered as a shortcut to the whole computation procedure when the knowledge of the exact solutions is not necessary, so that one could skip orthogonalization and normalization procedures along with the further summation over the polarization states to get the density matrix. Nevertheless, this approach could be of great benefit if one would apply it to problems with even higher number of degrees of freedom, therefore greatly reducing the computation time. It is worth noting that this approach heavily relies on the knowledge of the appropriate δ-function decomposition. If the latter is not known then the task becomes as hard as constructing the propagator from the exact solutions, because δ-function decomposition implicitly carries the correct normalization.

References

[1] Itzykson C, Zuber J B Quantum Field Theory (McGraw-Hill Inc., 1980)
[2] Kuznetsov A and Mikheev N *Electroweak Processes in External Active Media* (Berlin, Heidelberg: Springer-Verlag, 2013)