Integrable $F(R)$ gravity cosmological models with an additional scalar field

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6th International Conference on Particle Physics and Astrophysics
30.11.2022

Introduction

- $F(R)$ is one of the ways to generalize GR:

$$S_{GR}=\frac{M_{\mathrm{Pl}}^2}{2}\int d^4x\sqrt{-g}R+S_m \longrightarrow S_F=\int d^4x\sqrt{-g}F(R)+S_m\,;$$

- This action leads to the following equations:

$$F'R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}F-(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box)F'=\frac{1}{2}T_{\mu\nu};$$

- There is a well-known procedure for describing $F(R)$ models in terms of equivalent models in Einstein frame (GR with a scalar field minimally coupled with gravity);
- The inverse is also true: some models of GR with minimally coupled to gravity scalar fields are equivalent to $F(R)$ gravity models.

Introduction

- Consider the following chiral cosmological model:

$$S_E = \int d^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \right.$$
$$\left. -\frac{\varepsilon_\psi}{2}K(\phi)g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - V(\phi)\right];$$

- By doing conformal transformation of metric, $g^{\mu\nu} = K(\phi)\tilde{g}^{\mu\nu}$, one gets the following action in Jordan frame:

$$S_J = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\mathrm{Pl}}^2}{2K}\tilde{R} - \frac{\tilde{g}^{\mu\nu}}{2K}\left[1 - \frac{3M_{\mathrm{Pl}}^2K_{,\phi}^2}{2K^2}\right]\partial_\mu\phi\partial_\nu\phi \right.$$
$$\left. -\frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{V}{K^2}\right].$$

Introduction

- If $2K^2 = 3M_{\mathrm{Pl}}^2 K_{,\phi}^2$, then the action won't have a kinetic term for ϕ. The function $K(\phi)$ then takes the following form:

$$K(\phi) = K_0 \mathrm{e}^{\kappa\phi},$$

where $\kappa = \pm\sqrt{2/3}/M_{\mathrm{Pl}}$ and $K_0 = \mathrm{const} > 0$;

- Introducing

$$F_{,\sigma}(\sigma) = \frac{M_{\mathrm{Pl}}^2}{2K}, \quad V(\phi(\sigma)) = \frac{M_{\mathrm{Pl}}^4}{4}\frac{\sigma F_{,\sigma} - F}{F_{,\sigma}^2},$$

we get

$$S_J = \int d^4x\sqrt{-\tilde{g}}\left[F_{,\sigma}(\sigma)(\tilde{R} - \sigma) + F(\sigma) - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi\right].$$

Introduction

- The resulting action corresponds to an $F(R)$ gravity model with a massless scalar field ψ:

$$S_F = \int d^4x\sqrt{-\tilde{g}}\left[F(\tilde{R}) - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi\right];$$

- If $V(\phi) = \Lambda = \mathrm{const} > 0$, then this action is an action of pure R^2 gravity model:

$$S_F = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\mathrm{Pl}}^4}{16\Lambda}\tilde{R}^2 - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi\right].$$

Solutions in Einstein frame

- In spatially flat FLRW metric,

$$ds^2 = -dt^2 + a_E^2(t)\left(dx^2 + dy^2 + dz^2\right) ,$$

we get the following equations:

$$3M_{\mathrm{Pl}}^2 H_E^2 = \frac{1}{2}\dot{\phi}^2 + \frac{\varepsilon_\psi}{2} K\dot{\psi}^2 + V,$$

$$2M_{\mathrm{Pl}}^2 \dot{H}_E + 3M_{\mathrm{Pl}}^2 H_E^2 + \frac{1}{2}\dot{\phi}^2 + \frac{\varepsilon_\psi}{2} K\dot{\psi}^2 = V,$$

$$\ddot{\phi} + 3H_E\dot{\phi} - \frac{\varepsilon_\psi}{2} K\dot{\psi}^2 + V_{,\phi} = 0,$$

$$\ddot{\psi} + 3H_E\dot{\phi} + \frac{K_{,\phi}}{K}\dot{\phi}\dot{\psi} = 0,$$

where $H_E \equiv \dot{a}_E/a_E$. In what follows we have $K(\phi) = K_0 e^{\kappa\phi}$, with $\kappa = -\sqrt{2/3}/M_{\mathrm{Pl}}$, and $V = \Lambda = \mathrm{const} > 0$.

Solutions in Einstein frame

- Equation for H_E:

$$\dot{H}_E + 3H_E^2 = \lambda,$$

  where $\lambda \equiv \Lambda/M_{\rm Pl}^2$;

- The solution is

$$H_E(t) = \sqrt{\frac{\lambda}{3}}\,\frac{1 - C\mathrm{e}^{-2\sqrt{3\lambda}\,t}}{1 + C\mathrm{e}^{-2\sqrt{3\lambda}\,t}},$$

  where C is a constant of integration.

Solutions in Einstein frame

- $C > 0$:

$$H_E(t) = \sqrt{\frac{\lambda}{3}}\,\tanh\left(\sqrt{3\lambda}\,(t - t_0)\right);$$

- $C < 0$:

$$H_E(t) = \sqrt{\frac{\lambda}{3}}\,\coth\left(\sqrt{3\lambda}\,(t - t_0)\right);$$

- $C = 0$:

$$H_E(t) = \sqrt{\frac{\lambda}{3}};$$

- $C = \pm\infty$:

$$H_E(t) = -\sqrt{\frac{\lambda}{3}};$$

- Here $t_0 = \ln(|C|)/(2\sqrt{3\lambda})$.

Solutions in Einstein frame

- Equation for ϕ:

$$\ddot{\phi} = -3H_E\dot{\phi} + \frac{K'_{,\phi}}{K}\left(3M_{\mathrm{Pl}}^2 H_E^2 - \frac{1}{2}\dot{\phi}^2 - \Lambda\right)$$
$$= -3H_E\dot{\phi} + 3\kappa M_{\mathrm{Pl}}^2 H_E^2 - \frac{\kappa}{2}\dot{\phi}^2 - \kappa M_{\mathrm{Pl}}^2\lambda;$$

- General solution:

$$\phi(t) = \frac{2}{\kappa}\ln\left[A\cos\left(\frac{\kappa M_{\mathrm{Pl}}}{\sqrt{6}}\arccos\left(\frac{1 - C\mathrm{e}^{-2\sqrt{3\lambda}\,t}}{1 + C\mathrm{e}^{-2\sqrt{3\lambda}\,t}}\right) + B\right)\right],$$

where A and B are constants of integration.

Solutions in Einstein frame

In an explicitly real form, we get:

- for $C > 0$:

$$\phi(t) = \frac{2}{\kappa}\ln\left[A\cos\left[\frac{\kappa M_{\mathrm{Pl}}}{\sqrt{6}}\arccos\left[\tanh\left(\sqrt{3\lambda}(t-t_0)\right)\right]+B\right]\right];$$

- for $C < 0$:

$$\phi(t) = \frac{2}{\kappa}\ln\left[A\tanh^n\left(\frac{\sqrt{3\lambda}}{2}(t-t_0)\right)+B\coth^n\left(\frac{\sqrt{3\lambda}}{2}(t-t_0)\right)\right],$$

  where $n = \kappa M_{\mathrm{Pl}}/\sqrt{6}$;

- Solutions for $\dot{\psi}$ are given by

$$\dot{\psi}(t) = \frac{C_\psi}{a_E^3(t)}e^{-\kappa\phi(t)},$$

  where C_ψ is a constant of integration.

Solutions in Einstein frame

- In Jordan frame, we have

$$ds^2 = -d\tilde{t}^2 + a_J^2(\tilde{t})\left(dx^2 + dy^2 + dz^2\right);$$

- One can get solutions for $H_J(\tilde{t})$ using results for $\phi(t)$ and $H_E(t)$:

$$a_J(\tilde{t}(t)) = \sqrt{K(\phi(t))}\,a_E(t),$$

$$H_J(\tilde{t}(t)) = \frac{1}{a_J}\frac{da_J}{d\tilde{t}} = \frac{1}{\sqrt{K(\phi(t))}}\left[H_E(t) + \frac{1}{2}\frac{d\ln K}{dt}(t)\right],$$

where

$$\tilde{t} = \int \sqrt{K(\phi(t))}\,dt;$$

- Note that the cosmic time in Einstein frame t is a parametric time in Jordan frame.

Solutions of $F(R)$ gravity equations

- In spatially flat FLRW metric,

$$ds^2 = -\,d\tilde{t}^2 + a_J^2(\tilde{t})\left(dx^2 + dy^2 + dz^2\right),$$

we get the following $F(R)$ gravity equations for $F(R) = F_0 R^2$:

$$18F_0\left(6H_J^2\dot{H}_J - \dot{H}_J^2 + 2H_J\ddot{H}_J\right) = \frac{\varepsilon_\psi}{4}\dot{\psi}^2,$$

$$6F_0\left(18H_J^2\dot{H}_J + 12H_J\ddot{H}_J + 9\dot{H}_J^2 + 2\dddot{H}_J\right) = -\frac{\varepsilon_\psi}{4}\dot{\psi}^2;$$

- These equations lead to

$$\dddot{H}_J + 9H_J\ddot{H}_J + 18H_J^2\dot{H}_J + 3\dot{H}_J^2 = 0.$$

Solutions of $F(R)$ gravity equations

- By multiplying the equation by ${\dot{H}_J}^2$ and factoring out certain expressions, we get

$$\left(\ddot{H}_J+3H_J\dot{H}_J\right)\left(2H_J\dddot{H}_J+6H_J^2\ddot{H}_J+12H_J\dot{H}_J^2\right)$$
$$=\left(2H_J\ddot{H}_J+6H_J^2\dot{H}_J-\dot{H}_J^2\right)\left(\dddot{H}_J+3H_J\ddot{H}_J+3\dot{H}_J^2\right),$$

or, equivalently,

$$\left(\ddot{H}_J+3H_J\dot{H}_J\right)\frac{d}{d\tilde{t}}\left[\dot{H}_J^2-2H_J\ddot{H}_J-6H_J^2\dot{H}_J\right]$$
$$=\left(\dot{H}_J^2-2H_J\ddot{H}_J-6H_J^2\dot{H}_J\right)\frac{d}{d\tilde{t}}\left[\ddot{H}_J+3H_J\dot{H}_J\right].$$

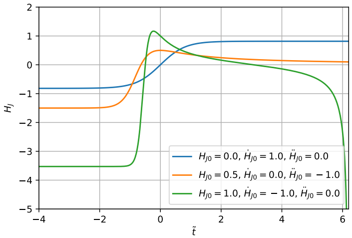
Solutions of $F(R)$ gravity equations

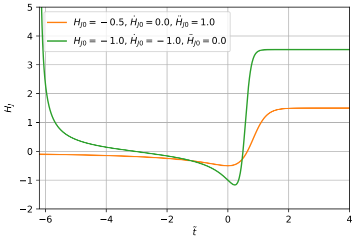
- Two cases:
  1. $\ddot{H}_J + 3H_J\dot{H}_J = 0$;
  2. $(C_1 + 2H_J)\ddot{H}_J + 3H_J(C_1 + 2H_J)\dot{H}_J - \dot{H}_J^2 = 0$.

  Here C_1 is a constant of integration.
- Both equations can be integrated:
  1. $2\dot{H}_J + 3H_J^2 = 2\tilde{C}$;
  2. $\dot{H}_J = C_2\sqrt{|C_1 + 2H_J|} + (C_1 + 2H_J)(C_1 - H_J)$.

  Here $\tilde{C}$ and C_2 are constants of integration.

Solutions of $F(R)$ gravity equations





Some solutions of the R^2 gravity equations. Note that the solutions in green correspond to the situation in which R changes its sign.

Solutions of $F(R)$ gravity equations

$\tilde{C}$	$H_J(\tilde{t})$	$\dot{\psi}(\tilde{t})$
$\tilde{C} > 0$, $\dot{H}_{J0} > 0$	$\sqrt{\frac{2\tilde{C}}{3}}\tanh\left(\sqrt{\frac{3\tilde{C}}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)$	$\frac{6\tilde{C}\sqrt{2F_0}}{\cosh^2\left(\sqrt{\frac{3\tilde{C}}{2}}(\tilde{t}-\tilde{t}')\right)}$
$\tilde{C} > 0$, $\dot{H}_{J0} < 0$	$\sqrt{\frac{2\tilde{C}}{3}}\coth\left(\sqrt{\frac{3\tilde{C}}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)$	$\frac{6\tilde{C}\sqrt{2F_0}}{\sinh^2\left(\sqrt{\frac{3\tilde{C}}{2}}(\tilde{t}-\tilde{t}')\right)}$
$\tilde{C} > 0$	$\frac{\sqrt{6\tilde{C}}\left(1-B\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}\right)}{3\left(1+B\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}\right)}$	$\frac{24B\tilde{C}\sqrt{2F_0}\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}}{\left(B\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}+1\right)^2}$
$\tilde{C} = 0$	$\frac{2}{3(\tilde{t}-\tilde{t}')}$	$\frac{4\sqrt{2F_0}}{(\tilde{t}-\tilde{t}')^2}$
$\tilde{C} < 0$	$-\frac{\sqrt{-6\tilde{C}}}{3}\tan\left[\frac{\sqrt{-6\tilde{C}}}{2}\left(\tilde{t}-\tilde{t}'\right)\right]$	$\frac{6\tilde{C}\sqrt{2F_0}}{\cos^2\left(\sqrt{\frac{-3\tilde{C}}{2}}(\tilde{t}-\tilde{t}')\right)}$

Conclusion

- We have found exact solutions of pure R^2 gravity model with a massless scalar field. In particular, we have solved both $F(R)$ gravity equations and equations of the equivalent model in Einstein frame.
- These results were published in our paper
  Vsevolod R. Ivanov and Sergey Yu. Vernov. Integrable modified gravity cosmological models with an additional scalar field. Eur. Phys. J. C, 81:985, 2021.