On quasinormal modes in 4D black hole solutions in the model with anisotropic fluid

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Overview

A certain family of **4D black hole solutions** is considered, which appear in the model with anisotropic fluid (H. Dehnen, V.D. Ivashchuk and V.N. Melnikov, 2003). This family is parametrized by a natural number $q \in \mathbb{N}$. For $q = 1$ the metric becomes the Reissner-Nordström one.

For this family of solutions:

- The *global causal structure* is presented
- Certain *physical parameters* corresponding to BH solutions (gravitational mass, PPN parameters, Hawking temperature and entropy) are calculated.
- The *quasinormal modes* (QNMs) are studied for a test massless scalar field in the eikonal approximation.
- The validity of the *Hod conjecture* (connecting the Hawking temperature and the damping rate) is investigated.

The model in $4D$ case

- Manifold: $\mathbb{R}_{(\text{radial})} \times \mathbb{S}^2 \times \mathbb{R}_{(\text{time})}$
- Stress-energy tensor: $(T^\mu_\nu) = \mathrm{diag}\,(p_r,\ p_t,\ p_t,\ -\rho)$
- Equation of state: $p_r = -\rho(2q-1)^{-1},\ p_t = -p_r, \quad \rho > 0, \quad q \in \mathbb{N}$
- Einstein equations: $R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \kappa T^\mu_\nu$
- The 4D solution takes the form $(c = 1)$:

$$\boxed{ds^2 = H^{2/q}\left[\frac{dr^2}{1-\frac{2\mu}{r}} + r^2 d\Omega_2^2 - H^{-4/q}\left(1-\frac{2\mu}{r}\right)dt^2\right]}$$

$$\kappa\rho = \frac{(2q-1)P(P+2\mu)(1-2\mu r^{-1})^{q-1}}{H^{2+\frac{2}{q}}\ r^4}$$

where $H \equiv H(r) := 1 + \dfrac{P}{2\mu}\left[1-\left(1-\dfrac{2\mu}{r}\right)^q\right]; \quad P, \mu > 0$

- *Special cases*:

$q = 1 \quad \Rightarrow$ RN with $Q^2 = P(P+2\mu)$, $r_g \sim GM = P + \mu$;

$q \to \infty \quad \Rightarrow$ Schwarzschild

Global structure

The global structure can be studied by analysing the behaviour of the "redshift function" $A(r)$ and "area function" $C(r)$ at critical points corresponding to horizons or singularities.

The metric: $ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + C(r)d\Omega^2$

- "Redshift function": $A(r) = (H^2(r))^{-1/q}\left(1 - \frac{2\mu}{r}\right)$
- "Area function": $C(r) = (H^2(r))^{1/q} r^2$

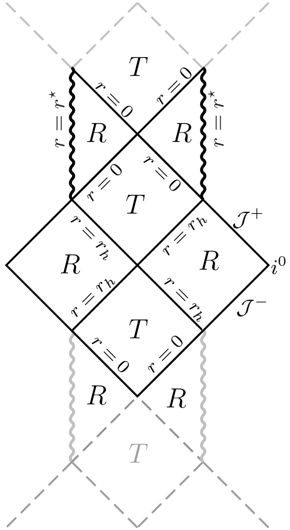
Denote by $r = r^\star$ the maximal root of the equation $H(r) = 0$.
We have $r^\star < 0$ for odd $q = 2k+1$ and $r^\star > 0$ for even $q = 2k$.

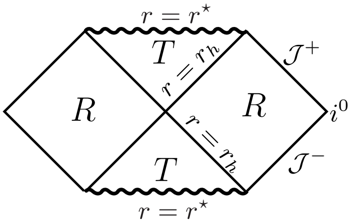
Critical points of the radial coordinate r:

1. $r = r_h \equiv 2\mu$ (regular external horizon)
2. $r = r^\star$ (the singularity)
3. $r = 0$ for odd $q = 2k+1$ (internal horizon)

Global structure: the Carter—Penrose diagrams

$q = 1, 3, 5, \ldots$	$q = 2, 4, 6, \ldots$





Global structure: extremal case

The extremal case: $\mu \to +0$.

$$ds^2 = H_e^{2/q} \left(dr^2 + r^2 d\Omega^2 - H_e^{-4/q} dt^2 \right), \quad \kappa\rho = \frac{(2q-1)P^2}{H_e^{2+\frac{2}{q}} \; r^4},$$

where $H_e \equiv H_e(r) = 1 + \dfrac{Pq}{r}$, with $P > 0$.

The scalar curvature: $R[g] = \dfrac{2(q-1)P^2}{(r+Pq)^{2+\frac{2}{q}} \; r^{2-\frac{2}{q}}}$.

- For $q = 1$ the metric is coinciding with the *extremal RN case* with "double" horizon at $r = +0$ and central singularity at $r = -P + 0$.
- For $q = 2, 3, 4, \ldots$ the metric describes a *naked singularity* at $r = 0$; $\lim\limits_{r \to +0} R[g] = +\infty$.

Gravitational mass and PPN parameters

Isotropic coordinates: $r = \bar{R}\left(1 + \frac{\mu}{2\bar{R}}\right)^2, \quad \bar{R}^2 \equiv \delta_{ij}x^i x^j, \quad i,j = 1,2,3$

The metric:

$$ds^2 = H^{2/q}\left[-H^{-4/q}\left(\frac{1-\frac{\mu}{2\bar{R}}}{1+\frac{\mu}{2\bar{R}}}\right)^2 dt^2 + \left(1+\frac{\mu}{2\bar{R}}\right)^4 \delta_{ij}dx^i dx^j\right]$$

Definition of PPN parameters:

$$g_{00} = -(1 - 2V + 2\beta V^2) + O(V^3),$$
$$g_{ij} = \delta_{ij}(1 + 2\gamma V) + O(V^2), \qquad V \equiv \frac{GM}{\bar{R}}$$

For our metric we get:

- $GM = \mu + P/q$
- $\beta - 1 = \dfrac{qA_f}{2(GM)^2}, \quad$ where $A_f = P(P + 2\mu)$
- $\gamma = 1$

Hawking temperature and entropy

Units: $c = \hbar = k_B = 1$.
The Hawking temperature of the black hole may be calculated using the well-known relation (J.W. York, 1985):

$$T_H = \frac{1}{4\pi\sqrt{-g_{00}g_{rr}}} \frac{d(-g_{00})}{dr}\Big|_{\text{horizon}}, \quad \text{where here } g_{rr} = A(r)^{-1}$$

For our metric we get:

- $T_H = \dfrac{1}{8\pi\mu}\left(1 + \dfrac{P}{2\mu}\right)^{-2/q}, \qquad q = 1, 2, \ldots$

The Bekenstein-Hawking (area) entropy $S = \mathcal{A}/(4G)$, corresponding to the horizon at $r = 2\mu$, where $\mathcal{A}$ is the horizon area, reads

- $S_{BH} = \dfrac{4\pi\mu^2}{G}\left(1 + \dfrac{P}{2\mu}\right)^{2/q}$

Motivation

- At present, the decaying oscillations such as quasinormal modes (QNMs) are very interesting and popular topic of research. A possible application of QNMs may be related to *gravitational waves (B.P. Abbott et al, 2016, 2019, 2020), emitted during the final ("ringdown") stage of binary black hole (BH) mergers*.
- It is believed that the frequencies of gravitational waves may be calculated by using certain superpositions of QNMs. The analysis of experimental data may clarify the nature of gravity in the regime of strong fields.
- Below we study QNMs for our 4D black hole solution in the eikonal approximation, and verify the validity of the Hod conjecture.

Quasinormal modes

For spherically symmetric and asymptotically flat BH solutions we seek the solutions to a wave equation in the form $\Phi(t,x) = e^{-i\omega t}\Phi_*(x)$, where $\Phi_*(x)$ obeys a Schrödinger-type equation [8, 9, 10, 11]:

$$\left(-\epsilon^2 \frac{d^2}{dx^2} + V(x)\right)\Phi_* = \omega^2 \Phi_*, \qquad \epsilon > 0 \text{ (typically } \epsilon = 1)$$

- $x \in (-\infty, +\infty)$ — tortoise coordinate
- Potential $V(x) > 0$, $\quad V(x) \to 0$ either when $x \to -\infty$ or $x \to +\infty$.
- ω — QNM frequency; typically $\mathrm{Re}(\omega) > 0$ and $\mathrm{Im}(\omega) < 0$

Analytical continuation method:

1. Start with the Schrödinger equation for a wave function $\Psi(x)$ of a particle "moving" in the potential $-V(x)$:
$$\left(-\hbar^2 \frac{d^2}{dx^2} - V(x)\right)\Psi = E\Psi$$
2. Find non-empty discrete spectrum $E_n = E(\hbar, n| - V)$, $n = 0, 1, \dots$
3. QNM frequencies are calculated as $\boxed{\omega^2 = -E(\hbar = i\epsilon, n| - V)}$. Here $n = 0, 1, \dots$ is an *overtone number*.

Quasinormal modes

Units: $\hbar = G = c = 1$.

- The Klein–Fock–Gordon equation for a test massless scalar field in the background metric $g_{\mu\nu}$:

$$\Delta\Psi \equiv \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Psi) = 0 \qquad (\mu,\nu = 0,1,2,3)$$

- General metric: $ds^2 = -A(u)dt^2 + B(u)du^2 + C(u)d\Omega^2$
- Separation of variables: $\Psi = e^{-i\omega t}e^{-\gamma}\Psi_*(u)Y_{lm}$,
  where $l = 0, 1, \dots$ (multipole quantum number), $m = -l, \dots, l$
- Equation for radial function $\Psi_*(u)$:

$$\boxed{\frac{d^2\Psi_*(u)}{du^2} + \left\{\frac{B}{A}\omega^2 - \frac{B}{C}l(l+1) - \gamma'' - (\gamma')^2\right\}\Psi_*(u) = 0}$$

where $\gamma = \dfrac{1}{2}\ln(B^{-1}C\sqrt{AB})$, $\gamma' = d\gamma/du$, $\gamma'' = d^2\gamma/du^2$

Quasinormal modes

Examine our BH solution for $r > 2\mu$ in terms of the "tortoise" coordinate r_* defined as a result of the transformation $dr_* = dr/A(r)$:

$$ds^2 = -Adt^2 + Adr_*^2 + Cd\Omega^2$$

In our solution:

- $A = H^{-a}\left(1 - \frac{2\mu}{r}\right), \quad C = H^a r^2 \equiv \exp(2\gamma), \qquad a = 2/q$
- $H = 1 + \frac{P}{2\mu}\left[1 - \left(1 - \frac{2\mu}{r}\right)^q\right] \equiv 1 + p(1 - z^q),$
  where $P,\ \mu > 0, \quad p \equiv P/(2\mu), \quad q \in \mathbb{N}$
- a new variable z is introduced:

$$z := 1 - \frac{2\mu}{r}, \quad r = \frac{2\mu}{1 - z}; \qquad 0 < z < 1 \text{ for } r > 2\mu$$

- $\gamma = \frac{1}{2}\ln C = \frac{1}{2}\ln(H^a r^2)$

Quasinormal modes

Choosing radial coordinate $u = r_*$, we have the equation for Ψ_*:

$$\frac{d^2\Psi_*}{dr_*^2} + \{\omega^2 - V\}\Psi_* = 0,$$

where ω is the (cyclic) *QNM frequency*; $V = V(r(r_*))$ is the effective potential: $V = \mathcal{V} + \delta\mathcal{V}$

- $\mathcal{V} = \frac{l(l+1)A}{H^a r^2} = \boxed{\frac{l(l+1)z(1-z)^2(1+p(1-z^q))^{-2a}}{(2\mu)^2}}, \qquad a = \frac{2}{q}.$

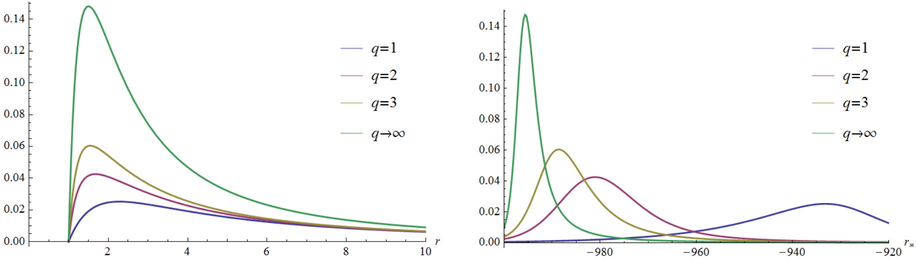
  This is an *eikonal part* of the effective potential.

- $\delta\mathcal{V} = \gamma'' + (\gamma')^2 = (\sqrt{C})''/\sqrt{C}$

  Notation for derivatives: $f' = \frac{df}{dr_*} = A\frac{df}{dr}$.

Quasinormal modes

Plots of the reduced potential $\dfrac{\mathcal{V}}{l(l+1)}$:



Plots of the reduced potential as a function of the radial coordinate r (left panel) and the tortoise coordinate r_* (right panel) for $P = 2\mu = 1$, $q = 1, 2, 3$ and the limiting case $q \to +\infty$.

The maximum of the effective potential is largest for $q \to +\infty$ and smallest for $q = 1$. At large distances the effective potential tends to zero, as expected.

Quasinormal modes

In what follows we consider *the eikonal approximation*: $\boxed{l \gg 1}$

The maximum of the eikonal part $\mathcal{V}$ is found from the extremum condition $\mathcal{V}' = A\dfrac{d\mathcal{V}}{dr} = 0$, which in terms of $z = 1 - 2\mu/r$ reads:

$$pz^{q+1} - 3pz^q + (1+p)(3z-1) = 0 \qquad (1)$$

Proposition 1. *For any $P, \mu > 0$ and $q \in \mathbb{N}$, this extremality relation has only one solution for $r > 2\mu$, which is the point of maximum for $\mathcal{V}(r)$.*

Denote this point of extremum by r_0. In terms of z we get that the point $\boxed{z_0 = 1 - 2\mu/r_0}$ is a unique solution to Eq. (1) for $z \in (0,1)$.

- The maximum of the eikonal part $\mathcal{V}$:

$$\boxed{\mathcal{V}_0 = \mathcal{V}(r_0) = \frac{l(l+1)}{H^{2a}(r_0)r_0^2}\left(1 - \frac{2\mu}{r_0}\right)}$$

Quasinormal modes

The second derivative of $\mathcal{V}$ with respect to the tortoise coordinate in the point of extremum:

$$\mathcal{V}_0'' = \left.\frac{d^2\mathcal{V}}{dr_*^2}\right|_{r_*=r_*(r_0)} = A_0^2 \left.\frac{d^2\mathcal{V}}{dr^2}\right|_{r=r_0} = A_0^2 \left(\frac{2\mu}{r_0^2}\right)^2 \left.\frac{d^2\mathcal{V}}{dz^2}\right|_{z=z_0}, \quad A_0 \equiv A(r_0)$$

After some algebra we have:

$$\boxed{\mathcal{V}_0'' = -\frac{1}{2} A_0^2 \left(\frac{2\mu}{r_0^2}\right)^2 \mathcal{V}_0 \mathcal{B}(z_0)},$$

where $\mathcal{B}(z) \equiv \frac{3}{2}q - \frac{2(q-2)z}{(1-z)^2}$.

Quasinormal modes

The square of the cyclic frequency in the eikonal approximation (E.Berti et al. 2009; R.A.Konoplya et al. 2011):

$$\omega^2 = \mathcal{V}_0 - i\left(n + \frac{1}{2}\right)\sqrt{-2\mathcal{V}_0''} + O(1), \qquad l \gg 1, \;\; l \gg n$$

We get the asymptotic relations (as $l \to +\infty$) on real and imaginary parts of ω in the eikonal approximation:

- $\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-a} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right),$
- $\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-a} \mu r_0^{-2} \mathcal{B}_0^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right),$

where $H_0 = H(r_0)$, $r_0 = 2\mu/(1 - z_0)$, $z_0 \in (0,1)$, and $\mathcal{B}_0 = \mathcal{B}(z_0)$.

Quasinormal modes

- Note that the parameters of the unstable circular null geodesics around stationary spherically symmetric and asymptotically flat black holes are in correspondence with the eikonal part of quasinormal modes of these black holes (V. Cardoso et al. 2009; M. Cvetič et al. 2016; R.A. Konoplya et al. 2017).
- Below we consider eikonal QNM for **three cases** $q = 1, 2, 3$ when the master equation (1) may be solved in radicals for all values of $p > 0$, and also in the limiting case $q = +\infty$.

In this case the master equation (1) is a quadratic one, and has the root

$$z_0 = z_0(1,p) = \frac{-3 + \sqrt{4p(p+1)+9}}{2p}; \qquad \forall p > 0 \quad z_0 \in (1/3, 1)$$

The eikonal QNMs may be rewritten as

- $\mathrm{Re}(\omega) \;=\; \left(l + \frac{1}{2}\right)\sqrt{\frac{\bar{M}}{\bar{r}_0^3} - \frac{Q^2}{2\bar{r}_0^4}} + O\left(\frac{1}{l+\frac{1}{2}}\right),$
- $\mathrm{Im}(\omega) \;=\; -\left(n + \frac{1}{2}\right)\sqrt{\frac{\bar{M}}{\bar{r}_0^3} - \frac{Q^2}{2\bar{r}_0^4}}\sqrt{\frac{3\bar{M}}{\bar{r}_0} - \frac{2Q^2}{\bar{r}_0^2}} + O\left(\frac{1}{l+\frac{1}{2}}\right),$

where $\bar{r}_0 = r_0 + P$, $\quad \bar{M} = \mu + P = GM$, $\quad Q^2 = 2P(P + 2\mu)$.

Note that $\bar{r}_0$ corresponds to the position of the unstable circular photon orbit in the RN spacetime:

$$ds^2 = -\bar{f}(\bar{r})dt^2 + \bar{f}(\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \qquad \bar{f}(\bar{r}) = 1 - \tfrac{2GM}{\bar{r}} + \tfrac{Q^2}{2\bar{r}^2}$$

For $n = 0$ it was obtained by N. Andersson et al. 1996.

Quasinormal modes. The case $q = 2$

In this case the master equation (1) is a cubic one, and has a real root

$$z_0 = z_0(2,p) = Z^{1/3} - p^{-1} Z^{-1/3} + 1, \quad \text{where} \;\; Z \equiv \frac{1}{p}\left(\sqrt{1+\frac{1}{p}} - 1\right).$$

The function $z_0(2,p)$ is monotonically increasing from $1/3$ to 1.

The eikonal QNMs read

- $$\mathrm{Re}(\omega) = \left(l+\frac{1}{2}\right) H_0^{-1} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l+\frac{1}{2}}\right),$$
- $$\mathrm{Im}(\omega) = -\left(n+\frac{1}{2}\right) H_0^{-1} \mu r_0^{-2}\sqrt{3} + O\left(\frac{1}{l+\frac{1}{2}}\right)$$

where $H_0 = 1 + \dfrac{P}{2\mu}\left[1 - \left(1 - \dfrac{2\mu}{r_0}\right)^2\right], \quad r_0 = \dfrac{2\mu}{1 - z_0}.$

In this case the master equation (1) is of fourth power, and has a real solution in radicals:

$$z_0 = z_0(3,p) = \frac{1}{2}\sqrt{X} - \frac{\sqrt{Y}}{4\sqrt{3}} + \frac{3}{4},$$

where

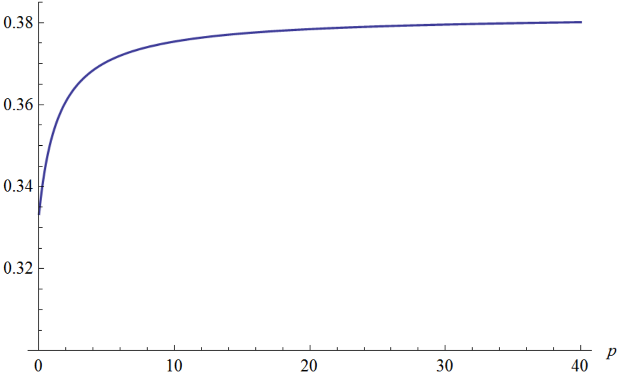
$$X = -\frac{3\sqrt{3}}{2}\left(1 - \frac{8}{p}\right)Y^{-1/2} - Z^{1/3} - \frac{5(p+1)}{3p}Z^{-1/3} + \frac{9}{2}$$

$$Y = 12Z^{1/3} + 27 + 20\left(1 + \frac{1}{p}\right)Z^{-1/3},$$

$$Z = \frac{p+1}{2p^2}(9 + 3^{-3/2}\sqrt{2187 - 500p(p+1)}),$$

Quasinormal modes. The case $q = 3$

It may be verified that $z_0 = z_0(3, p)$ is real for all $p > 0$ and obey $1/3 < z_0 < \frac{3-\sqrt{5}}{2} \approx 0,382$.



The graphical representation of the function $z_0 = z_0(3, p)$.

Relations for QNMs in this case read as follows

- $$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-2/3} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right),$$
- $$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-2/3} \mu r_0^{-2} \left(\frac{9}{2} - \frac{2 z_0}{(1 - z_0)^2}\right)^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right),$$

where $H_0 = 1 + \dfrac{P}{2\mu}\left[1 - \left(1 - \dfrac{2\mu}{r_0}\right)^3\right]$, $r_0 = \dfrac{2\mu}{1 - z_0}$, and $z_0 = z_0(3, p)$.

Quasinormal modes. The case $q = +\infty$

In the case $q \to +\infty$ the relations for QNMs yield

- $$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right)\sqrt{\frac{\mu}{r_0^3}} + O\left(\frac{1}{l + \frac{1}{2}}\right),$$
- $$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right)\sqrt{\frac{\mu}{r_0^3}} + O\left(\frac{1}{l + \frac{1}{2}}\right),$$

where $r_0 = 3\mu = 3GM$ corresponds the position where the BH effective potential attains its maximum.

We note that $r_0 = 3\mu$ is the radius of the photon sphere for the Schwarzschild BH with the metric

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right)dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

which is coinciding with limiting case of our BH solution when $q = +\infty$. These relations for Schwarzschild spacetime were obtained by H.J. Blome et al. 1984.

Remark. Here we restrict our choice of a test field by a massless (spin-zero, non-charged) scalar field which is the simplest "perturbation" to study.

It may be shown that the consideration of a *test Maxwell field* on our black hole background will lead us to two equations on functions: $\Psi_{*,a} = a_{lm}(r_*)$ and $\Psi_{*,b} = b_{lm}(r_*)$, which are certain combinations of coefficients (and their derivatives) coming from decomposing of vector potential in (vector) spherical harmonics. These equations (one of them is just an integrability condition) look like the radial equation for Ψ_* considered above, but with another potential $V = \mathcal{V}$(and $\delta V = 0$ in this case).

Thus, one can obtain a similar spectrum of QNM in the eikonal approximation for a test Maxwell field as for a massless scalar field considered here.

The Hod conjecture

Here we verify the conjecture by S. Hod, 2007 on the existence of QNMs obeying the inequality

$$\boxed{|\mathrm{Im}(\omega)| \leq \pi T_H}$$

where T_H is the Hawking temperature.

- The Hod conjecture has been tested in theories with higher curvature corrections such as the Einstein-Dilaton-Gauss-Bonnet and Einstein-Weyl for the Dirac (with positive result) field (A.F. Zinhailo, 2019).
- Recently, the Hod conjecture was also verified (with positive result) for a dyon-like dilatonic BH solution for certain values of dimensionless parameter $a \in [0, 1]$ (A.N. Malybayev, K.A. Boshkayev, V.D. Ivashchuk, 2021).

The Hod conjecture

Here we verify this conjecture by using the obtained eikonal relations for $\mathrm{Im}(\omega)$ and the relation for the Hawking temperature T_H.

For our purpose it is sufficient to check the validity of the inequality

$$\boxed{y < 1},$$

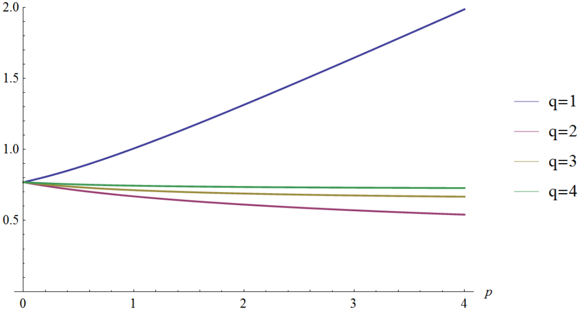
where

$$y = y(p,q) \equiv \frac{|\mathrm{Im}(\omega_{\mathrm{eik}})(n=0)|}{\pi T_H}$$
$$= \left(\frac{1+p}{1+p(1-z_0^q)} \right)^{2/q} \times (1-z_0)^2 \sqrt{\frac{3}{2}q - \frac{2(q-2)z_0}{(1-z_0)^2}}$$

We use the limiting "eikonal value" given by the first term in $\mathrm{Im}(\omega)$ for the lowest overtone number $n = 0$.

- **Proposition 2.** *The dimensionless parameter $y = y(p,q)$ obeys the inequality $y < 1$ for all $p = P/\mu > 0$ and $q \in \{2, 3, 4, \dots\}$.*

- The Proposition 2 has been proved for $q > 2$ analytically (for $q = 2$ verified numerically).
- This result can be illustrated by a plot of the function $y(p,q)$ for a particular set of values of q:



The graphical representation of the function $y(p,q)$ for $q = 1, 2, 3, 4$.

Remark. Let us comment also on the case $q = 1$ which gives us the RN metric. It may be readily verified that in this case *the inequality* $y < 1$ *is not satisfied* for all values of p: it is valid only for $0 < p < p_{cr}$, where p_{cr} is some critical value of parameter p (A.N. Malybayev, K.A. Boshkayev, V.D. Ivashchuk, 2021).

As it was pointed out, the violation of the Hod inequality in the eikonal regime for certain p (and $n = 0$) does not close the possibility for the obeying this relation for exact values of QNM for certain $l = 0, 1, 2, \dots$ and all values of parameter p.

We also note that recently some examples of the violation of the Hod conjecture have been discussed for certain black hole solutions in supergravity and other theories (M. Cvetič, G.W. Gibbons, C.N. Pope, 2016).

Conclusion I

- We have studied a family of non-extremal BH solutions in a 4D gravitational model with anisotropic fluid proposed by H. Dehnen, V.D. Ivashchuk and V.N. Melnikov, 2003 [1]. The equations of state for the fluid contains a natural parameter q.
- For $q = 1$ the metric of our solution coincides with the RN metric, while in the limit $q = +\infty$ it becomes the Schwarzschild metric.
- We have outlined the **global (causal) structure** of these solutions: for odd $q = 2k + 1$ the Carter-Penrose diagram is of RN type, while for even $q = 2k$ it is of of Schwarzschild type.
- Certain **physical parameters** for our BH solutions are presented: the gravitational mass M, the Hawking temperature, BH area entropy.

Conclusion II

- We have examined the solutions to massless Klein-Fock-Gordon equation in the background of our static BH metric. By using the tortoise coordinate r_* we have reduced this equation to the radial one governed by a certain **effective potential** V containing the parameters of solution $P, \mu > 0$, $q \in \mathbb{N}$, and the multipole quantum number $l = 0, 1, \ldots$.
- We have studied the eikonal part $\mathcal{V}$ of the effective potential for large $l \gg 1$ and obtained an **algebraic master equation** for the value $z_0 = 1 - 2\mu/r_0$, where r_0 is the value of the radial coordinate corresponding to the maximum of the eikonal part.
- By using the maximum value of the eikonal part of the potential, we have calculated **the cyclic frequencies of the QNMs in the eikonal approximation** up to a solution of the master equation in z_0.

Conclusion III

- For obtained values of eikonal QNMs we have also considered **special cases** $q = 1, 2, 3$ and a limiting cases $q = +\infty$. For $q = 1$ our eikonal relations are compatible with the well-known result for RN solution for $n = 0$, while for $q = +\infty$ they in an agreement with the well-known result for the Schwarzschild solution.
- We have also verified **the Hod conjecture** for our BH solutions by considering eikonal QNMs frequences with the lowest value of the overtone number $n = 0$. It is shown that *the Hod conjecture is valid in the range of* $q > 1$.
- We note that the results for eikonal QMN modes of test massless (non-charged) scalar field are also valid for some other test fields, e.g. for electromagnetic one.

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Thank you
for your attention!

Appendix A. Proof of the Proposition 1. I

Here we prove the Proposition 1. It is equivalent to the following Lemma.
Lemma. *For any $p > 0$ and $q \in \mathbb{N} = \{1, 2, 3 \dots\}$, the master equation*

$$pz^{q+1} - 3pz^q + (1+p)(3z-1) = 0 \tag{A.1}$$

has only one solution $z_0 = z_0(p,q)$, belonging to interval $(0,1)$. This solution obeys the inequality

$$\mathcal{B}(z_0) = \frac{3}{2}q - \frac{2(q-2)z_0}{(1-z_0)^2} > 0 \tag{A.2}$$

for all $p > 0$ and $q \in \mathbb{N}$.
Proof. Since $z = 1/3$ is not a solution to eq. (A.1) we present the master equation in the following form

$$F(z) = F(z,q) = z^q \frac{z-3}{3z-1} = -b = -1 - \frac{1}{p} < 0, \tag{A.3}$$

$p > 0$. The functions $F(z) = F(z,q)$, $q = 1, 2, 3, 4$, are presented at the figure below.

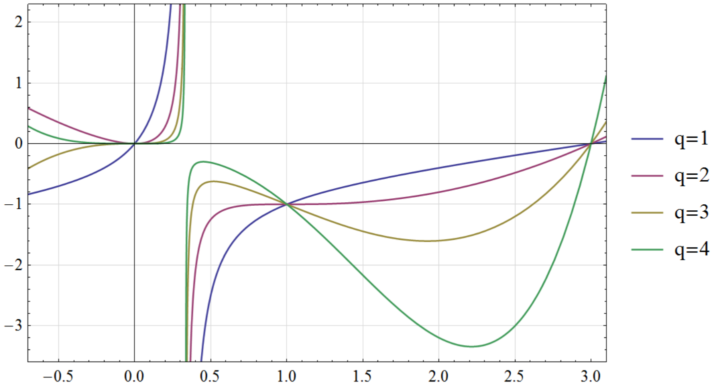

Figure: The graphical representation of the functions $F(z) = F(z, q)$ for $q = 1, 2, 3, 4$.

It follows from the definition (A.3) that

$$F(z,q) > 0, \tag{A.4}$$

for $z \in (0, 1/3)$, $q \in \mathbf{N}$ and

$$\lim_{z \to 1/3 \pm 0} F(z,q) = \mp\infty, \tag{A.5}$$

$$\lim_{z \to 1-0} F(z,q) = -1, \tag{A.6}$$

for all $q \in \mathbf{N}$. Hence the seminterval $(0, 1/3]$ should be excluded in our search the solution to Eq. (A.3).
Let us analyze behavior of the function $F(z) = F(z,q)$ for $z \in (1/3, 1)$ and fixed $q \in \mathbf{N} = \{1, 2, 3 \dots\}$. The first derivative reads

$$\frac{dF(z)}{dz} = \frac{\partial F(z,q)}{\partial z} = z^{q-1} \frac{[3qz^2 + (8 - 10q)z + 3q]}{(3z-1)^2}. \tag{A.7}$$

For $\boxed{q = 1, 2}$, we have $\frac{dF(z)}{dz} > 0$ for $z \in (1/3, 1)$ and hence the function $F(z)$ is monotonically increasing from $-\infty$ to -1, when $z \in (1/3, 1)$. By applying the Intermediate Value Theorem to our continuous monotonically increasing function $F(z) = F(z, q)$, $q = 1, 2$, we get that for any $p > 0$ there exist unique $z_0(p, q) \in (0, 1)$, with $z_0(p, q) > 1/3$, which obeys eq. (A.1). Inequality (A.2) is obviously satisfied for $q = 1, 2$. That means that the Lemma is valid for $q = 1, 2$.

Now we consider the case $\boxed{q > 2}$. From (A.7) we obtain that the there exists a unique point of extremum of the function $F(z, q)$ in the interval $(1/3, 1)$

$$z_1 = z_1(q) = \frac{10q - 8 - \sqrt{(16q - 8)(4q - 8)}}{6q}, \tag{A.8}$$

$1/3 < z_1(q) < 1$, which is the first root of the quadratic equation $3qz^2 + (8 - 10q)z + 3q = 0$. The second root $z_2(q) = 1/z_1(q) \in (1, 3)$ is irrelevant for our consideration.
The calculations give us: $z_1(3) = (11 - 2\sqrt{10})/9 \approx 0,5195$, $z_1(4) = (4 - \sqrt{7})/3 \approx 0,4514$, $z_1(5) = (7 - 2\sqrt{6})/5 \approx 0,4202$ and $F(z_1(3)) \approx -0,6227$, $F(z_1(4)) \approx -0,2987$, $F(z_1(5)) \approx -0,1297$. We note that

$$z_1(q+1) < z_1(q), \tag{A.9}$$

for all $q > 2$. This follows from monotonical decreasing of the function $z_1(q)$ for $q > 2$, since $z_1(q) = 1/z_2(q)$ and

$$z_2(q) = \frac{10 - 8/q + \sqrt{(16 - 8/q)(4 - 8/q)}}{6}, \tag{A.10}$$

is monotically increasing in q for $q > 2$.

It may be verified that

$$z_1(q) \to \frac{1}{3}, \qquad F(z_1(q)) \to 0, \tag{A.11}$$

for $q \to +\infty$. Indeed, it follows from (A.8) that

$$z_1(q) = \frac{1}{3} + \frac{1}{3q} + O(q^{-2}), \tag{A.12}$$

and

$$F(z_1(q)) \sim \frac{1}{3^q}\left(1 + \frac{1}{q}\right)^q \left(-\frac{8}{3}\right) q \sim -\frac{8e}{3^{q+1}} q \to 0 \tag{A.13}$$

as $q \to +\infty$.

The function $F(z) = F(z,q)$ (for $q > 2$) is monotonically increasing in the interval $(1/3, z_1)$, since $\frac{dF(z)}{dz} > 0$ in this interval, see (A.7), while it is monotonically decreasing in the interval $(z_1, 1)$ due to inequality $\frac{dF(z)}{dz} < 0$ which is valid there. Hence we get

$$F(z_1(q), q) > F(z, q) > F(1, q) = -1 \tag{A.14}$$

for all $z \in (z_1, 1)$ and $q > 2$. This implies that the semi-interval $[z_1(q), 1)$ should be excluded in our search of solution to equation (A.3) for a given $q > 2$. Thus, we restrict our consideration to $z \in (1/3, z_1(q))$.
Let us define $z_*(q) \in (1/3, z_1(q))$, which obeys the following equation

$$F(z_*(q), q) = -1, \tag{A.15}$$

$q > 2$. By applying the Intermediate Value Theorem for a continuos monotonically increasing function $F(z(q), q)$ defined on $(1/3, z_1(q))$ and using (A.5) and (A.14) one can readily prove that such point does exist and is unique for any $q > 2$.

The calculations give us

$$z_*(3) = \frac{3-\sqrt{5}}{2} \approx 0,382, \quad z_*(4) \approx 0,346, \quad z_*(5) \approx 0,337. \tag{A.16}$$

It may be proved that

$$z_*(q+1) < z_*(q), \tag{A.17}$$

for any natural $q > 2$. Indeed, if we suppose that $z_*(q+1) \geq z_*(q)$ for some q we get from monotonical increasing of the function $F(z, q+1)$ in $(1/3, z_1(q+1))$ and obvious inequality $F(z, q+1) > F(z, q)$ for $z \in (1/3, 1)$ that

$$-1 = F(z_*(q+1), q+1) \geq F(z_*(q), q+1) > F(z_*(q), q) = -1$$

and hence we come to a contradiction. Thus, the chain of inequalities (A.17) is correct.
Now we return to our original equation (A.3). From monotonical increasing of the function $F(z,q)$ in $(1/3, z_1(q))$ we get that $F(z) \geq F(z_*(q)) = -1$ for $z \in [z_*(q), z_1(q))$ and hence the semi-interval $[z_*(q), z_1(q))$ should be excluded for our consideration of (A.3). By applying once more the Intermediate Value Theorem for a continuos monotonically increasing function $F(z(q), q)$ defined on $(1/3, z_*(q))$ and using (A.5) and (A.15) we can find that the point z_0 which obeys the equation (A.3) does exist, belongs to $(1/3, z_*(q))$ and is unique for any $q > 2$ and $p > 0$. We denote this point as $z_0 = z_0(p,q)$. Thus, we have

$$1/3 < z_0(p,q) < z_*(q) < z_1(q), \tag{A.18}$$

for all $q > 2$ and $p > 0$. It follows from (A.11) and (A.18)

$$z_0(p,q) \to \frac{1}{3}, \tag{A.19}$$

as $q \to +\infty$ uniformly in $p \in (0, +\infty)$.
We note that one can present the solution as

$$z_0(p,q) = F_q^{-1}\left(-1 - \frac{1}{p}\right), \tag{A.20}$$

where F_q^{-1} is the function which is inverse to the function $F_q : (1/3, z_*(q)) \longrightarrow (-\infty, -1)$, defined as $F_q(z) = F(z,q)$. The function F_q^{-1} is a continuos and monotonically increasing one (due to a proper theorem on inverse function). It may be readily verified that

$$\lim_{p \to +\infty} z_0(p,q) = z_*(q), \tag{A.21}$$

and

$$\lim_{p \to +0} z_0(p,q) = 1/3. \tag{A.22}$$

Thus, the first part of the Lemma is proved for all $q \in \mathbf{N}$. Now, we should prove the second part of the Lemma for $q > 2$ (for $q = 1, 2$ it was checked above). Let us consider the function

$$\mathcal{B}(z) = \frac{3}{2}q - \frac{2(q-2)z}{(1-z)^2} \tag{A.23}$$

for $z \in (0,1)$ and $q = 3, 4, \dots$. We get

$$\mathcal{B}(z) = \frac{3qz^2 + (8-10q)z + 3q}{2(1-z)^2} = \frac{3q(z - z_1(q))(z - z_2(q))}{2(1-z)^2}, \tag{A.24}$$

where $z_1(q) < 1$ and $z_2(q) > 1$ are defined by relations (A.8) and (A.10), respectively. We find that $\mathcal{B}(z) > 0$ for all $z \in (0, z_1(q))$ and hence for $z = z_0(p,q)$ with $q > 2$ and $p > 0$. We remind that $1/3 < z_0(p,q) < z_*(q) < z_1(q)$ for all $q > 2$ and $p > 0$. Thus, the inequality (A.2) is satisfied. **The Lemma is proved**.

First we consider the case $q > 2$. In what follows we use the relation

$$\frac{1}{3} < z_0 = z_0(p,q) < 0.4, \tag{B.1}$$

for all $p > 0$ and $q > 2$. Indeed, it follows from relations (A.16), (A.17) and (A.18) given at Appendix A that

$$\frac{1}{3} < z_0 = z_0(p,q) < z_*(q) \leq z_*(3) = \frac{3-\sqrt{5}}{2} \approx 0,382, \tag{B.2}$$

for all $p > 0$ and $q \geq 3$. Thus, relation (B.1) is correct.
In what follows we use the following splitting

$$y = y_1 y_2 y_3, \tag{B.3}$$

$$y_1 = \left[\tfrac{1+p}{1+p(1-z_0^q)}\right]^{2/q}, \qquad y_2 = (1-z_0)^2, \quad y_3 = \sqrt{\mathcal{B}(z_0)}, \tag{B.4}$$

where $\mathcal{B}(z) = \frac{3}{2}q - \frac{2(q-2)z}{(1-z)^2}$.

For y_1 we obtain from (B.1)

$$y_1 = y_1(p,q) = \left[\frac{1}{1-\frac{p}{p+1}z_0^q}\right]^{2/q} < \left[\frac{1}{1-z_0^q}\right]^{2/q} < \left[\frac{1}{1-(0.4)^q}\right]^{2/q}, \tag{B.5}$$

for all $p > 0$ and $q > 2$. Now, we use the following fact about the function

$$\tilde{f}(q) = \left[\frac{1}{1-u^q}\right]^{2/q}, \tag{B.6}$$

where $0 < u < 1$ and $q > 0$. Namely, the function $\tilde{f}(q)$ is monotonicall decreasing in $(0,+\infty)$. This follows from the relation

$$\frac{d\tilde{f}(q)}{dq} = \tilde{f}(q)\frac{2}{q^2(1-x)}\left[(1-x)\ln(1-x) + x\ln x\right] < 0, \tag{B.7}$$

where $x = u^q$ and $0 < x < 1$. This fact imlies for $u = 0.4$ the following bound

$$y_1 = y_1(p,q) < \left[\frac{1}{1-(0.4)^q}\right]^{2/q} \leq \left[\frac{1}{1-(0.4)^3}\right]^{2/3} \approx 1.04507975. \tag{B.8}$$

for all $p > 0$ and $q \geq 3$. Hence, we get

$$y_1 = y_1(p,q) < 1.0451, \tag{B.9}$$

for all $p > 0$ and $q > 2$.
For y_2 we obtain from (B.1)

$$y_2 = y_2(p,q) = (1-z_0)^2 < \frac{4}{9}, \tag{B.10}$$

for all $p > 0$, $q > 2$.

The last bound

$$y_3 = y_3(p,q) = \sqrt{\mathcal{B}(z_0)} < \sqrt{\mathcal{B}(1/3)} = \sqrt{3}, \tag{B.11}$$

is also valid for all $p > 0$ and $q > 2$. It follows from monotonical decreasing of the function $\mathcal{B}(z)$ in $(0,1)$ and $1/3 < z_0 < z_* < z_1$. Here $\mathcal{B}(z) > 0$ for $z \in (0, z_1)$ and $z_* = z_*(q)$, $z_1 = z_1(q)$ are defined in Appendix A.

Plugging the bounds (B.9), (B.10), (B.11) into (B.3) we find

$$y = y(p,q) < 1.0451 \times (4/9) \times \sqrt{3} \approx 0.804518, \tag{B.12}$$

and hence

$$y = y(p,q) < 0.80452 < 1, \tag{B.13}$$

for all $p > 0$, $q > 2$. **The proposition 2 is proved.**