Wormholes in a Friedmann universe

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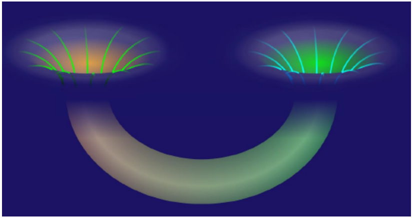

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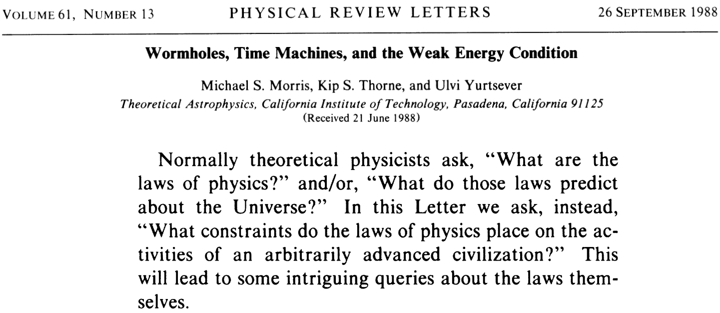
Plan of the talk

- Wormholes in general. Motivation
- Spherically symmetric space-times. Possible dynamic wormholes
- Static wormholes, the necessity of NEC violation
- Dynamic wormholes, different definitions of a throat
- Possible dynamic wormholes without NEC violation
- Lemaître-Tolman-Bondi (LTB) solution with a magnetic field
- Possible throats in LTB space-times
- Matching LTB and Friedmann space-times
- Wormholes in a Friedmann universe: numerical estimates
- Concluding remarks



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Wormholes, Time Machines, and the Weak Energy Condition

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Normally theoretical physicists ask, "What are the laws of physics?" and/or, "What do those laws predict about the Universe?" In this Letter we ask, instead, "What constraints do the laws of physics place on the activities of an arbitrarily advanced civilization?" This will lead to some intriguing queries about the laws themselves.

Wormhole physics has become quite a popular research area, even though nobody has ever seen a real wormhole in Nature. Thus, on November 29, 2022, a search for the term "wormhole" on the site arxiv.org gave 2,465 results for all years and 183 results for past 12 months. This popularity looks natural since a wormhole is one more, in addition to a black hole, and even simpler manifestation of a strongly curved space-time which can lead to many effects of great interest both in fundamental physics and astrophysics. To mention a few, these are opportunities of time machines and fast travels, a possible relation to quantum entanglement and potential signatures in astronomical observations.

A **wormhole:** a tunnel or shortcut between different universes or distant regions of the same universe (a Lorentzian, two-way traversable wormhole).
(We leave aside quantum or Euclidean wormholes, as well as "nontraversable" or "one-way traversable" wormholes with horizons = black holes.)

Let us, for simplicity, assume **spherical symmetry** (although other kinds of wormholes are also widely discussed):

$$ds^2 = e^{2\gamma(x,t)}dt^2 - e^{2\alpha(x,t)}dx^2 - r^2(x,t)d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

where x is a radial coordinate, and $r(x,t)$ is the spherical radius.

Definition. A **throat** in a spatial section $t = \text{const}$ is a regular minimum of the function $r(x,t)$.

Definition. A **wormhole** is a space-time region containing at least one throat with $r = r_{\rm th}$ and radii $r(x,t) \gg r_{\rm th}$ on both sides of the throat(s).

These definitions directly extend those used for static metrics. For dynamic ones there is ambiguity in the choice of spatial sections, and there are some other definitions (*Hochberg and Visser (1998), Hayward (1999), etc*) that rest on the behavior of null congruences. In this work, however, there are physically preferable spatial sections related to the comoving reference frame, and the above definitions are suitable and intuitively clear.

For **static wormholes**, the above definition naturally refers to the static reference frame, in which the difference $\binom{0}{0} - \binom{1}{1}$ of the Einstein equations in suitable coordinates reads

$$\frac{r''}{r} \sim -(\rho + p_r) \qquad (\rho\text{=density, } p_r\text{=radial pressure})$$

On a throat $r' = 0$, $r'' > 0$, hence $\rho + p_r < 0 \;\Rightarrow\;$ NEC violation, "exotic matter".

Dynamic wormholes defined as above are attached to a specific set of spatial sections of a spherically symmetric space-time, they, in general, exist in a finite time interval, but do not require NEC violation and are not excluded in space-times with normal, non-exotic matter.

A regular minimum requires $r' = \partial r/\partial x = 0$. On the other hand, the equality

$$r^{,\alpha} r_{,\alpha} \equiv g^{tt}\dot{r}^2 + g^{xx}r'^2 = \mathrm{e}^{-2\gamma}\dot{r}^2 - \mathrm{e}^{-2\alpha}r'^2 = 0$$

indicates an apparent horizon, so that r may be used as a time coordinate if $r^{,\alpha} r_{,\alpha} > 0$ (a *T-region*) and as a spatial one if $r^{,\alpha} r_{,\alpha} < 0$ (an *R-region*). Thus a throat, where $r' = 0$, gets into a T-region if $\dot{r} \neq 0$. This excludes many opportunities.

For example, in a dynamic asymptotically flat space-time we cannot obtain a wormhole since a possible throat can be located only beyond a horizon, hence it is a black hole. If both $r' = 0$ and $\dot{r} = 0$ (e.g., in static metrics), a horizon or a throat are both possible, it depends on the properties of g_{tt}.

Matter = neutral dust $\Rightarrow$ comoving reference frame, the metric

$$ds^2 = d\tau^2 - \mathrm{e}^{2\lambda(R,\tau)}dR^2 - r^2(R,\tau)d\Omega^2, \qquad (1)$$

with τ = proper time along particle trajectories,
R = radial coordinate (specified up to $R \mapsto F(R)$),
$r(R,\tau)$ = time-dependent radius of the Lagrangian sphere "number" R.
SET of dustlike matter: only $T_0^0 = \rho \neq 0$.

Radial (external) electromagnetic field: $T_\mu^{\nu[\mathrm{em}]} = \dfrac{q^2}{8\pi G r^4}\,\mathrm{diag}(1,\ 1,\ -1,\ -1)$,
$q = \mathrm{const}$ = magnetic charge (for certainty) .
Einstein equations, $R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R \delta_\mu^\nu \Lambda = -8\pi G T_\mu^\nu \quad \Rightarrow$

$$2r\ddot{r} + \dot{r}^2 + 1 - e^{-2\lambda}r'^2 = \frac{q^2}{r^2} + \Lambda r^2, \qquad (2)$$

$$\frac{1}{r^2}(1 + \dot{r}^2 + 2r\dot{r}\dot{\lambda}) - \frac{\mathrm{e}^{-2\lambda}}{r^2}(2rr'' + r'^2 - 2rr'\lambda') = 8\pi G\rho + \frac{q^2}{r^4} + \Lambda, \qquad (3)$$

$$\dot{r}' - \dot{\lambda}r' = 0. \qquad (4)$$

Conservation law:

$$\dot{\rho} + \rho\Big(\dot{\lambda} + \frac{2\dot{r}}{r}\Big) = 0, \quad \Rightarrow \quad \rho = \frac{F'(R)}{r^2 r'}, \quad F(R) = \text{arbitrary function}. \qquad (5)$$

Integrate Eq. (4) in $\tau \quad \Rightarrow$

$$e^{2\lambda} = \frac{r'^2}{1+f(R)}, \qquad f(R) = \text{arbitrary function}, \ f(R) > -1. \tag{6}$$

Using that, integrate Eq. (2) in $\tau \quad \Rightarrow$

$$\dot r^2 = f(R) + \frac{F(R)}{r} - \frac{q^2}{r^2} + \frac{\Lambda}{3} r^2, \tag{7}$$

where $F(R)$ is the same as in (5). According to (5),

$$F(R) = 2GM(R) = 8\pi G \int \rho r^2 r' dR \quad (M(R) = \text{mass function}). \tag{8}$$

$f(R) \mapsto$ initial velocity distribution. At $\Lambda = 0$, from (7) it follows:

$f > 0$ — hyperbolic motion, making possible $r(R,\tau) \to \infty$.
$f = 0$ — parabolic motion,
$f < 0$ — elliptic (finite) motion.

At $\Lambda \neq 0$, things are more involved, the boundary of finite motion is shifted.

Further integration of Eq. (7) with $\Lambda \neq 0$ leads to elliptic integrals. In what follows we assume $\Lambda = 0$, so that only elementary functions are necessary.

Throats in LTB space-times ($\Lambda = 0$)

A throat is, by our definition, a regular minimum of $r(R,\tau)$ at fixed τ. In terms of a well-behaved radial coordinate R ($\mathrm{e}^{\lambda} > 0$), we have $r' = 0$, $r'' > 0$.
According to (6), $\mathrm{e}^{-2\lambda} r'^2 = 1 - f(R)$, hence on the throat $R = R_{\mathrm{th}}$

$$\mathrm{e}^{-2\lambda} r'^2 = 1 + f(R_{\mathrm{th}}) = 0 \quad \Rightarrow \quad f(R_{\mathrm{th}}) = -1, \quad \text{or} \quad h(R_{\mathrm{th}}) = 1, \qquad (9)$$

where $h(R) = -f(R)$. It immediately follows that **only elliptic models with $h(R) > 0$ are compatible with throats.** To keep the metric (1) nondegenerate, it must be in general $1 + f = 1 - h > 0 \quad \Rightarrow \quad h = 1$ occurs at a single point $R = R_{\mathrm{th}}$, so that at $R = R_{\mathrm{th}}$ we have a maximum of $h(R)$: $h'(R_{\mathrm{th}}) = 0$, $h''(R_{\mathrm{th}}) < 0$.

Calculation of r'' from (6) leads to $r''\Big|_{R=R_{\mathrm{th}}} = -\dfrac{h'}{2r'}\Big|_{R=R_{\mathrm{th}}} > 0.$

A finite density $\rho > 0$ requires $F'/r' > 0 \;\Rightarrow\; F'$ should vanish together with r'.

We summarize that on a regular throat $R = R_{\mathrm{th}}$ with $\rho > 0$ it holds

$$r' = 0, \qquad h = 1, \qquad h' = 0, \qquad h'' < 0; \qquad \frac{h'}{r'} < 0;$$
$$F' = 0, \qquad \frac{F'}{r'} > 0, \qquad F^2 - 4hq^2 > 0. \qquad (10)$$

(the last inequality also follows from the equations of motion.)

Further integration of (7) in τ for elliptic models gives

$$\pm[\tau - \tau_0(R)] = \frac{1}{h}\sqrt{-hr^2 + Fr - q^2} + \frac{F}{2h^{3/2}} \arcsin \frac{F - 2hr}{F^2 - 4hq^2}, . \qquad (11)$$

the arbitrary function $\tau_0(R)$ $\mapsto$ clock synchronization at different R. A well-known convenient parametrization of the solution (for $\tau_0(R) \equiv 0$):

$$\begin{aligned} r &= \frac{F}{2h}(1 - \Delta\cos\eta), \qquad \Delta = \sqrt{1 - \frac{4hq^2}{F^2}}, \qquad 0 < \Delta \leq 1, \\ \tau &= \frac{F}{2h^{3/2}}(\eta - \Delta\sin\eta). \end{aligned} \qquad (12)$$

At $\Delta = 1$ (LTB solution with **pure dust**), there are singularities $r = 0$ at $\eta = 0, 2\pi$, etc., while at $q \neq 0$, $\Delta < 1$, the value $r = 0$ is never achieved.

At $q=0$ $(\Delta=1)$, under the assumptions (e.g., *Landau-Lifshitz, Field Theory*)

$$F(\chi) = 2a_0 \sin^3\chi, \qquad h(\chi) = \sin^2\chi, \qquad a_0 = \mathrm{const} \qquad (13)$$

(using the radial coordinate $R = \chi$ of angular nature), the solution describes Friedmann's closed isotropic universe filled with dust, such that

$$r = r(\eta, \chi) = a(\eta)\sin\chi, \qquad a(\eta) = a_0(1 - \cos\eta) = \text{scale factor}. \qquad (14)$$

At a throat, the quantity $-\lim_{R\to R_{\rm th}} [Fh'/(F'h)] = B > 0$ is finite. Then we obtain

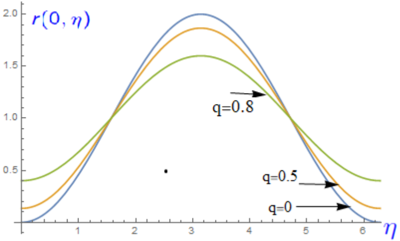
$$r'\Big|_{R_{\rm th}} \approx \frac{F'(2N_2 - BN_1)}{4\Delta\,(1-\Delta\,\cos\eta)}, \qquad (15)$$

$$\rho\Big|_{R_{\rm th}} = \frac{F'}{8\pi G r^2 r'}\Big|_{R_{\rm th}} = \frac{\Delta(1-\Delta\,\cos\eta)}{2\pi G r^2(2N_2 - BN_1)}\Big|_{R_{\rm th}}. \qquad (16)$$

$$N_1(R,\eta) = \cos\eta - 3\Delta + 3\Delta^2(\cos\eta + \eta\sin\eta) + \Delta^3(-2+\cos^2\eta),$$

$$N_2(R,\eta) = -\cos\eta + 2\Delta - \Delta^2(\cos\eta + \eta\sin\eta) \qquad (17)$$

Example: $h(R) = \dfrac{1}{1+R^2}, \;\; F(R) = b(1+R^2), \;\; b = \mathrm{const}$ (length scale)



Time dependence of the throat radius $r(0,\eta)$ for $b = 1$, $q = 0, 0.5, 0.8$. The singular value $r = 0$ is reached only if $q = 0$.

There are also shell-crossing singularities close to $\eta = 0$ and $\eta = 2\pi$ connected with $N_* = 2N_2 - BN_1 \to 0$, where $r \neq 0$, $r' \to 0$, and $\rho \to \infty$. Thus even with $q \neq 0$ the wormholes have a finite lifetime. It is quite a general property of all such models.

For space-time regions with different dust distributions, separated by a particular value of R, the well-known Darmois-Israel junction conditions lead to the requirements

$$[r] = 0, \quad [\mathrm{e}^{-\lambda} r'] = 0 \quad \Rightarrow \quad [f] = 0, \quad [F] = 0. \tag{18}$$

In other words, to match two particular solutions, we must simply identify the values of $F(R)$ and $h(R)$ on the junction surface $R = R_{\rm th}$. The parameters q, Λ should also coincide on both sides of $R = R_{\rm th}$.

If $\rho = 0 \iff F = \mathrm{const}$ in a certain region, then it is (electro)vacuum with the Reissner-Nordström-(anti) de Sitter metric written in geodesic coordinates and comoving to a certain set of electrically neutral test particles. By matching with it, we can obtain models of evolving finite dust configurations along with their exteriors.

Further on, we will consider matching of LTB regions representing wormholes to other special LTB regions, namely, isotropic cosmological models.

Suppose that a wormhole region is matched to a Friedmann universe described by Eqs. (13), (14). Hence, we are dealing with $q = 0$.

To obtain a particular model, we must choose $h(R)$ and $F(R)$. We take into account: **(a)** the throat conditions (10), **(b)** the freedom of choosing the R coordinate, and **(c)** the assumption of wormhole symmetry with respect to its throat $(R = 0)$. We can make **without loss of generality** a particular choice of $h(R)$; then, the choice of $F(R)$ will be significant.

We choose: $\boxed{h(R) = \frac{1}{1+R^2}, \quad F(R) = 2b(1+R^2)^k,} \quad b, k = \mathrm{const} > 0.$

Friedmann: $\boxed{h(\chi) = \sin^2\chi, \quad F(\chi) = 2a_0 \sin^3\chi.}$

If the two metrics are matched at $\chi = \chi_* \ll 1$, then, putting $\sin\chi_* \approx \chi_*$,

$$1 + R_*^2 = \chi_*^{-2}, \qquad F(R_*) = 2b\chi_*^{-2k}, \qquad \frac{b}{a_0} = \chi_*^{2k+3}. \tag{19}$$

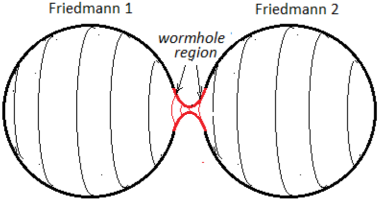
We also have in the wormhole solution $r(R,\eta) = b(1+R^2)^{k+1}(1-\cos\eta)$, therefore the constant b can be used as the upper estimate of the wormhole throat radius $r_{\rm th} = r(0,\eta)$.

The table shows some estimates, assuming $a_0 = 10^{28}\,\mathrm{cm} \approx$ size of the observed part of the Universe and $b = 10\,\mathrm{m}$, 10 000 km, or 1 pc.

Parameter	$b = 10$ m, $k = 1$	$b = 10$ m, $k = 0.1$	$b = 10^4$ km $k = 0.1$	$b = 1$ pc, $k = 0.1$
χ_*	10^{-5}	1.5×10^{-8}	1.1×10^{-6}	1.05×10^{-3}
r_*	30 kpc	50 pc	4 kpc	3.5 Mpc
$\rho_*,\ \mathrm{g/cm^3}$	2.2×10^{-30}	2.2×10^{-31}	2.2×10^{-31}	2.2×10^{-31}
$\rho_{\rm th},\ \mathrm{g/cm^3}$	$2.2\ \times10^{20}$	2.2×10^{19}	2.2×10^{7}	2.5×10^{-12}

The wormhole region has the size of parsecs or larger even for small throats.

Near the throat, the density is $\rho_{\rm th} \gg \rho_{\rm nuclear}$ for $b = 10\,\mathrm{m}$.
It is of white-dwarf order near a throat of planetary size, and quite small near a throat of 1 pc.



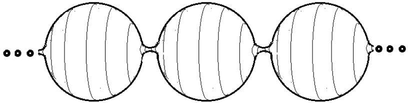
At the junction, the density ρ_* is of mean cosmological order of magnitude.

Small values of Λ cannot qualitatively change such local phenomena as the existence of wormholes. However, the global evolution with $\Lambda > 0$ must lead to accelerated expansion, which must also encompass the wormhole region.

Inclusion of small $q \neq 0$ also cannot strongly change the local picture of a wormhole. However, globally:

(i) The Universe cannot be precisely homogeneous and isotropic due to a vector field;

(ii) A charge on one "pole" inevitably leads to an opposite charge on the other, where the lines of force again converge. There can be a similar wormhole and one more universe beyond it, and so on. The whole picture will resemble a "churchhella", the wonderful Georgian dessert.



Also, two mouths of a wormhole may open in a single universe. A still more complex picture emerges if this happens with a few wormholes.

Concluding remarks

- Dynamic wormholes with finite lifetime, evolving together with the whole Universe, are possible without exotic matter.
- We have constructed models of dynamic wormholes on the basis of the LTB solution of general relativity, connecting two closed Friedmann universes filled with dust, or two distant parts of the same universe.
- Wormhole throats of macroscopic or even planetary size require very large matter densities. The size of a wormhole region matched to cosmology is a few orders of magnitude larger than that of the throat.
- Inclusion of Λ makes it possible to obtain such wormholes in a closed universe with accelerated expansion, or in the ΛCDM model (closed version). Constructuon of such models is a problem for future studies.
- The possible magnetic field of evolving wormholes may substantially complicate the global geometry and topology.

Reference: Kirill A. Bronnikov, Pavel E. Kashargin, and Sergey V. Sushkov, *Magnetized dusty black holes and wormholes,* Universe 7, 419 (2021); arXiv: 2109.12570

THANK YOU!