Spectrum of quantum BTZ black hole formed by a collapsing dust shell

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Introduction

- Solutions to classical General Relativity contain singularities in which the theory looses predictability.
- Can quantum theory resolve these singularities?

Quantization of a collapsing dust shell in 3+1 dimensions

V.Berezin (1990's-2000's)

- Spherically symmetric metrics

$ds^2 = -g_{00}dt^2 + g_{01}drdt + g_{11}dr^2 + R^2 d\Omega^2$

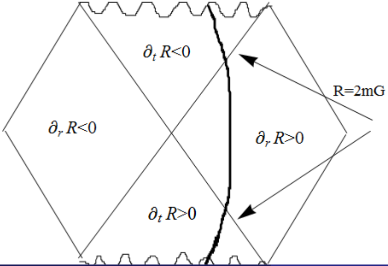
- Phase space reduction within ADM formalism

$g_{ij}(x), \pi^{ij}(x) \to P_R, R, m \neq M_{bare}, (T?)$

- Hamiltonian constraint

$$C = \left(1 - \frac{2mG}{R}\right) + 1 - 2\sqrt{1 - \frac{2mG}{R}} \cosh\left(\frac{GP_R}{R}\right) - \frac{M_{bare}^2 G^2}{R^2}$$

- Define the wavefunction on the entire Penrose diagram:



Quantization of a collapsing dust shell in 3+1 dimensions (continued)

- Wavefunction is an analytic function defined on a two-fold Riemann surface, as $\sqrt{1-\frac{2mG}{R}}$ is not single–valued.
- Define the bypassing rules for the branching point $R = 2mG$
- WDW equation is not a differential equation but an equation in finite differences. A first hint on discreteness

Stationary Solutions

$$\Psi(m, R^2 + i\zeta) + \Psi(m, R^2 - i\zeta) = \frac{\left(1-\frac{2mG}{R}\right) + 1 - \frac{M_{bare}^2 G^2}{R^2}}{\sqrt{1-\frac{2mG}{R}}}\Psi(m, R^2),$$

where $\zeta = m_{Pl}^2/(2m^2)$ $\longrightarrow$ mass spectrum

- .....but there is a parametrization of the entire phase space of the model by real variables. $\to$ spectrum of radius
  $\to$ dynamics

BTZ Black Hole

- The BTZ black hole, Bañados, Teitelboim, Zanelli 1992, in "Schwarzschild" coordinates is described by the metric

$$ds^2 = -(N)^2 dt^2 + N^{-2} dr^2 + R^2 d\phi, \tag{1}$$

with lapse function

$$N = \left(1 - 2m + \frac{R^2}{\ell^2}\right)^{1/2}. \tag{2}$$

(From now on we use units in which Newton's constant $G = 1$) The parameters m is the ADM mass, which is related to the mass M in original BTZ conventions as $M = 2m - 1$.

- The metric 1 satisfies the ordinary vacuum field equations of (2+1)-dimensional general relativity with a cosmological constant $\Lambda = -1/\ell^2$.
- BTZ black holes are locally isometric to anti-de Sitter space ADS^2.

Action principle

- The basic variable is SO(2,2)-connection A_μ^{AB}, where $A, B = 0..3$. Here $A_\mu^{3a} = e_\mu^a / l$ is the triad, where $l = 1/\sqrt{\Lambda}$, and $A_\mu^{ab} = \omega_\mu^{ab}$ is the Lorentzian connection, where $a, b = 0..2$.
- The total action consists of gravity action in the Chern-Simons form and the shell action

$$S = \frac{l}{8\pi} \int_M d^3x \epsilon^{\mu\nu\rho} \langle A_\mu, (\partial_\nu A_\rho + \frac{2}{3} A_\nu A_\rho) \rangle + S_{shell}, \tag{3}$$

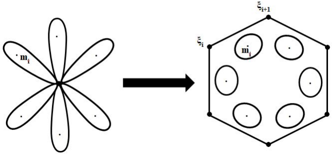
where $A_\mu = \Gamma_{AB} A_\mu^{AB}$ is so(2,2) connection, and $\langle , \rangle$ is a bilinear form on so(2,2) algebra, the Newton constant G is taken to be 1.
- The shell is discretized (represented as an ensemble of N particles)

$$S_{shell} = \sum_i^N \int_{l_i} Tr(K_i A_\mu) dx^\mu, \tag{4}$$

where l_i is i-th particle worldline and $K_i = m_i \Gamma_{03}$ – a fixed element of so(2,2)-algebra, M_i is the mass of i-th particle.

Phase space reduction

- Cut spacetime into N regions (discs) each containing one particle and an outer region (polygon), containing no particles (Alekseev, Malkin)



- Apply the results of 't Hooft, Matschull, Welling for each particle: solve the constraints, plug the solution back into the action. The symplectic form collapses to the vertices of the polygon:

$$\Omega_i = d\langle e^{-K_i}(\delta g_i g_i^{-1}) e^{K_i} \wedge \delta g_i g_i^{-1} \rangle. \tag{5}$$

- For cylindrically symmetric arrangement of the particles, the sum of the symplectic form for each particle is combined into a single form

$$\Omega_{full} = \langle \delta g_0 g_0^{-1}, \wedge U^{-1} \delta U \rangle, \quad U = \prod_i g_i^{-1} e^{K_i} g_i \tag{6}$$

Momentum space

- The Lorenzian part of holonomy U provides a global chart for the entire momentum space. It is a rotation outside the horizon and a boost inside. Its geometry is ADS^2.

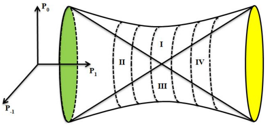


Figure: ADS-momentum space and its four regions. $(p_{-1}, p_0, p_1$ are coordinates of three dimensional flat space in which ADS^2 is embedded)

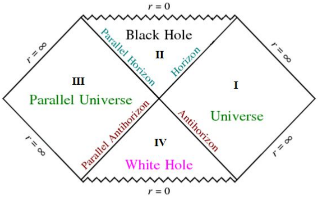


Figure: Corresponding four regions on the Penrose diagram

- It satisfies the constraint $TrU = \cos(\sqrt{1-2m})$, where m is the total mass. This is the Hamiltonian constraint

so(2,2)-algebra as a classical Drinfeld double.

- Lorentz transformations generated by $J^a = \epsilon^{abc}\Gamma_{ab}$, translations by $P_a = \Gamma_{a3}$,$[J^a, J^b] = \epsilon^{abc} J_c$, $[P^a, P^b] = \Lambda\epsilon^{abc} J_c$. Translations do not form a subalgebra
- Choose a basis $x_0 = 2iJ_1$, $x_1 = -J_0 + iJ_2$, $x_2 = J_0 + iJ_2$ $X_0 = \frac{1}{2}iP_1$, $X_1 = \frac{1}{2}(P_0 + iP_2) + \frac{\Lambda}{2}x_2$, $X_1 = \frac{1}{2}(-P_0 + iP_2) - \frac{\Lambda}{2}x_1$, with possible exchange $J_1, P_1 \rightleftarrows J_0, P_0$.
- Now Lorentz transformations form $sl(2)$ subalgebra : $[x_0, x_1] = 2x_1$, $[x_0, x_2] = -2x_2$, $[x_1, x_2] = x_0$
  Modified translations also form a subalgebra $[X_0, X_1] = \frac{\Lambda}{2}X_1$, $[X_0, X_2] = -\frac{\Lambda}{2}X_2$, $[X_1, X_2] = 0$, which is a sum of two Borel subalgebras of $sl(2)$, $B^+ \oplus B^-$ with diagonal elements identified.
- Cross commutation relations between new translations and Lorentz transformations leave Ad-invariant the following bilinear form

$$\langle x_a, X^b\rangle = \delta_a^b, \ \langle x_a, x^b\rangle = 0, \ \langle X_a, X^b\rangle = 0 \tag{7}$$

- This algebra is classical Drinfeld double $D(sl(2))$

so(2,2)-algebra as a classical Drinfeld double.

- This can be promoted to a Lie bialgebra with cocommutator given by

$$\delta_D(Y) = [Y \otimes 1 + 1 \otimes Y, r], \ \forall Y \in \{x_a, X_a\}, \tag{8}$$

where

$$r = \sum_a X_a \otimes x^a \tag{9}$$

is the classical r-matrix. It automatiacally satisfies classical Yang-Baxter equation. cocommutator depends on its skew symmetric part,

$$r' = \sum_a X_a \wedge x^a \tag{10}$$

In terms of the initial generators, J_a, P^a it can be rewritten as

$$r' = \underbrace{(\Lambda) J_0 \wedge J_2}_{skew\ \ sym.\ part\ \ of\ \ sl(2)\ \ r-matrix} + \underbrace{-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2}_{survives\ \ in\ \ \Lambda \to 0\ \ limit} \tag{11}$$

Poisson-Lie structure on the phase space and classical DD.

- On the phase space of the shell a symplectic form has been derived

$$\Omega_{shell} = \langle \delta h_0 h_0^{-1}, \wedge U^{-1}\delta U\rangle = \langle e^K \delta h_0 h_0^{-1} e^{-K}, \wedge \delta h_0 h_0^{-1}\rangle \quad (12)$$

where h_0 is $SO(2,2)$ transformation between a point on the shell and the origin, K is a Lorentz generator which leaves singularity worldline stable, and $U = h_0 e^K h_0^{-1}$ - the holonomy around the shell.

- Decompose $h_0 = h_L h_T$, where $h_L = \exp(\alpha_a x^a)$ - Lorentz transform and $h_T = \exp(\beta_a X^a)$ -modified translation which is a subgroup. then

$$\Omega_{shell} = \langle \delta h_T h_T^{-1}, \wedge U_L^{-1}\delta U_L\rangle = \langle \delta h_T h_T^{-1}, \wedge h_L^{-1} e^{-K} \delta h_L h_L^{-1} e^K h_L\rangle$$

Poisson brackets

$$\{h_T, \otimes U_L\} = (1 \otimes U_L) r (h_T \otimes 1), \quad (13)$$

with r-matrix from the previous slide.

- The infenitesimal version of this Poisson-Lie group is $D(sl(2))$ Lie bialgebra, and its quantization results in quantum double $D(SL_q(2))$.

Quantum double: coordinate and momentum space are both non-linear and non-commutative.

- Deformation of the algebra of observables with $q = \exp(-\pi\sqrt{|\Lambda|}\hbar)$ or $q = \exp(i\pi\sqrt{|\Lambda|}\hbar)$.
- Quantum double $D(SL_q(2))$ is a unity of quantum universal enveloping algebra, $U_q(sl(2))$, and its dual, quantized algebra of functions on a group, $Fun(SL_q(2))$, with commutation relations between the two.
- Coordinate space is the algebra of deformed translations in ADS^3 space: $U_q(sl(2))$: $X_\pm, H$,

$$q^{H/2} X_\pm q^{-H/2} = q^{\pm 1} X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{14}$$

Casimir: $C_2 = X^+ X^- + \left(\frac{q^{\frac{1}{2}(H-1)} - q^{-\frac{1}{2}(H-1)}}{q - q^{-1}} \right)^2$

Quantum double (continued).

- Momentum space is an $SL_q(2)$ holonomy around the shell

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Non-commutative algebra $Fun(SL_q(2))$: a, b, c, d, $ad - qbc = 1$

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc$$

$$bc = cb, \qquad ad - da = (q - q^{-1})bc$$

- Cross commutation relations between $U_q(sl(2))$ and $Fun(SL_q(2))$ $(a \leftrightarrow c, b \leftrightarrow d)$

$$q^H a = q^{-1} a q^H, \quad q^H b = q b q^H, \quad X_- a = q^{1/2} a X_-, \quad X_+ b = q^{-1/2} b X_+$$

$$X_+ a = q^{1/2} a X_+ + q^{-1/2} b q^{H/2}, \quad X_- b = q^{-1/2} b X_- + q^{1/2} a q^{H/2}$$

Worldline of the singularity and *-relations

- $so(2,2)$: Lorentz generators $J_0 = -J_0^*$, $J_{1,2} = J_{1,2}^*$, and translation generators $P_0 = -P_0^*$, $P_{1,2} = P_{1,2}^*$.
- Depending on the total energy the shell collapses either to point particle with trajectory along P_0, or to BTZ black hole along P_1.
- Point particle: timelike singularity

$$\Omega_{shell} = \langle e^{iJ_0}\delta h_0 h_0^{-1} e^{-iJ_0}, \wedge\delta h_0 h_0^{-1}\rangle \tag{15}$$

$D(SU_q(1,1))$-case, $H = iP_0$, q-real

$$a^* = d, \quad b^* = qc, \quad H^* = H, \quad X_\pm^* = -X_\mp, \quad q^* = q$$

- Black hole: spacelike singularity

$$\Omega_{shell} = \langle e^{iJ_1}\delta h_0 h_0^{-1} e^{-iJ_1}, \wedge\delta h_0 h_0^{-1}\rangle \tag{16}$$

$D(SL_q(2))$-case, $H = iP_1$,q-root of unity

$$a^* = a, \quad b^* = b, \quad H^* = -H, \quad X_\pm^* = -X_\pm, \quad q^* = q^{-1}$$

Momentum representation

- The states are ordered polynomials acting on $\mathbf{1}$
$$\Psi = \sum \alpha_{klmn} a^k b^l c^m d^n \mathbf{1} \tag{17}$$
If $k, l, m, n \geq 0$ these states are not normalizible (as it could be seen in $q \to 1$ limit). These states correspond to finite dimensional non-unitary representations of $sl(2)$.
Normalizible state need to contain negative degrees of combinations of a, b, c, d which are invertible.
- In $SU_q(1,1)$ case $aa* = ad = 1 + bb* \geq 1$, so a is invertible and the same for d.
- The lowest weight states are
$$\Psi_{n,n} = a^{-n}\mathbf{1}, \quad \Psi_{n,-n} = d^{-n}\mathbf{1} \tag{18}$$
The rest is obtained by applying $X_\pm$ to them. The time operator $T = H$ has discrete, but unbounded spectrum, $T\Psi_{n,n} = n\Psi_{n,n}$. The structure of representations is the same as in $q = 1$ limit. But this is a no black hole case

Coordinate operators and their spectra

- In $SL_q(2)$ case the invertible combination is $\tilde{a} = q^{1/2}a - ib + ic + q^{-1/2}d$, $\tilde{a}^*\tilde{a} = q + q^{-1} + a^2 + b^2 + c^2 + d^2 > 1$
- $\tilde{a}\mathbf{1}$ is an eigenstate of $\tilde{H} = i(q^{-1/2}X_+q^{-H/2} - q^{+1/2}X_-q^{-H/2})$, $\tilde{H}\tilde{a}\mathbf{1} = \tilde{a}\mathbf{1}$. By $q \to 1$-correspondence this is time operator $T = \tilde{H}$, it is hermitian $T^* = T$.
- The lowest weight normalizible states are:

$$\Psi_{l,l} = \prod_{k=1}^{l} \left(q^k(q^{1/2}a + ic) + (-ib + q^{-1/2}d) \right)^{-1} \mathbf{1}$$

$$\Psi_{l,-l} = \prod_{k=1}^{l} \left(q^k(q^{-1/2}a - ic) + (ib + q^{1/2}d) \right)^{-1} \mathbf{1}$$

- The eigenvalues of time operator

$$T\Psi_{l,\pm l} = \frac{q^{\pm l} - q^{\mp l}}{q - q^{-1}} \Psi_{l,\pm l} = [\pm l]_q \Psi_{l,\pm l}$$

Unlike $SU_q(1,1)$, the eigenvalues of time operator are now q-integers.

Coordinate operators and their spectra (continued)

- Other states can be derived without explicit construction of $X_\pm$

$$\Psi_{l-n,l+n} = \Psi_{l,l} \prod_{k=0}^{n-1} \Big(-q^{-l-k}(q^{1/2}a + ic) + (-ib + q^{-1/2}d) \Big) \mathbf{1}$$

$$\Psi_{l-n,-l-n} = \Psi_{l,-l} \prod_{k=0}^{n-1} \Big(-q^{-l-k}(q^{-1/2}a - ic) + (ib + q^{1/2}d) \Big) \mathbf{1}$$

here $0 \leq n < l$.

- Eigenstates of T and C_2:

$$T\Psi_{l,n} = [n]_q \Psi_{l,n}, \tag{19}$$

$$C_2\Psi_{l,n} = ((q^{\frac{1}{2}(l-1)} - q^{-\frac{1}{2}(l-1)})/(q - q^{-1}))^2 \Psi_{l,n}$$

$$l = 0...N, \quad n = -N.. - l, l..N, \quad N = 1/(\sqrt{|\Lambda|}\hbar) \quad q^N = -1 \tag{20}$$

Inside the BH n and l vary within a finite range, $\rightarrow$ Hilbert space is finite-dimensional

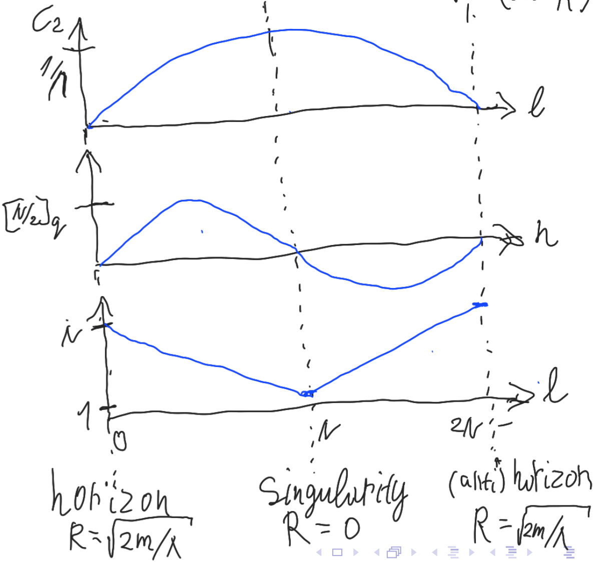
Coordinate operators and their spectra (pictures)

- The shell radius in terms of Casimir $R = \sqrt{2m\left(C_2 - \frac{1}{\lambda}\right)}$

Casimir eigenvalues

Time eigenvalues

Dimensionality of representation



Quantization (dynamics)

- The quantum version of the Hamiltonian constraint is a finite difference equation

$$\psi(t-1,\tilde{b}\tilde{c})+\psi(t+1,\tilde{b}\tilde{c})=H(\tilde{b}\tilde{c})\psi(t,\tilde{b}\tilde{c}), \qquad (21)$$

where This is Klein-Gordon-like equation for discrete time.

- Its Schroedinger version is $\psi(t+1,\tilde{b}\tilde{c})=\mathbf{U}(\tilde{b}\tilde{c})\psi(t,\tilde{b}\tilde{c})$, where $\mathbf{U}=H+\sqrt{H-1}$ – evolution operator.

$$H=\frac{\cos(\pi Q)}{1+\tilde{b}\tilde{c}}, \quad b=M\frac{\sinh\bar{\chi}\sin(\pi Q)}{\pi Q}, \qquad (22)$$

where

$$Q=\sqrt{\left(1-M\sqrt{\cosh^2(\bar{\chi})}\right)^2-\left(M\sinh(\bar{\chi})\right)^2}. \qquad (23)$$

and $\bar{\chi}$ is a parameter to be excluded.

- U is bounded, Ψ exponentially decrease at large momenta. Matrix elements of U must be everywhere finite

Near horizon transitions $q \to 1$

The transition amplitude from outside the horizon to inside the horizon, I $\to$ II, and back, II $\to$ I, in one step in time is calculated numerically.

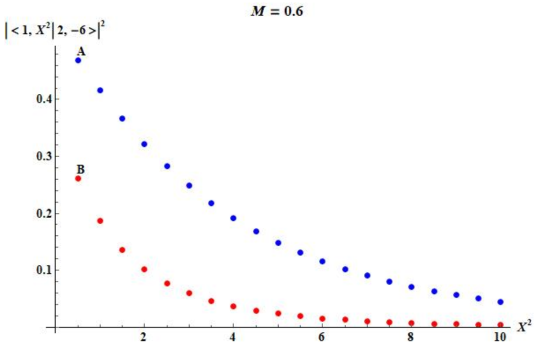


Figure: II $\to$ I (curve B) vs. I $\to$ II (curve A) relative transition rate.

II $\to$ I transition rate is non-zero, but exponentially damped away from the horizon.

Conclusions

- The Hilbert space of the shell inside the black hole is finite-dimensional, the spectrum of the shell radius is discrete and bounded
- One can argue that transition amplitudes between different shell radii, including $R = 0$ singularity are everywhere finite
- The shell bounces off the singularity

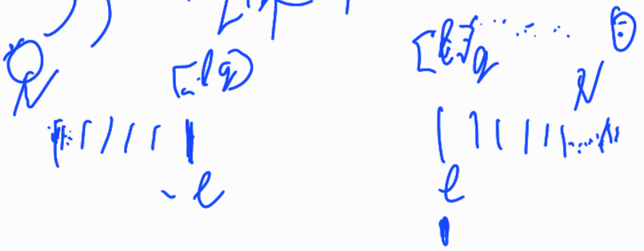
$s l 2$

1) Неунитарн. конечномерн.

$n = -l \ldots l$

2) унитарн. ∞ мерн.

3) $|q| = 1$



$[n\ q]$