Cornwall-Jackiw-Tomboulis effective action in (2+1)-dimensional models

Roman N. Zhokhov
IZMIRAN, IHEP
6th International Conference on Particle Physics and Astrophysics

Russian Science Foundation
Фонд развития теоретической физики и математики

K.G. Klimenko, IHEP

T.G. Khunjua, University of Georgia, MSU

details can be found in

Phys.Rev.D 106 (2022) 8, 085002 arXiv:2208.11735 [hep-th]
Phys.Rev.D 105 (2022) 2, 025014 arXiv: 2112.13012 [hep-th]
Int.J.Mod.Phys.A 36 (2021) 31n32, 2150231

The work is supported by

- Russian Science Foundation (RSF)

  under grant number 19-72-00077

- Foundation for the Advancement of Theoretical Physics and Mathematics

Lagrangian of Gross-Neveu model

$$L = \overline{\psi}_k \gamma^\nu i \partial_\nu \psi_k + \frac{G}{2N} \left(\overline{\psi}_k \psi_k \right)^2$$

$k = 1, ..., N$ is a number of flavours

It exhibits chiral symmetry breaking and dynamical mass generation

Note that the definition of chiral symmetry is slightly unusual in (2+1)-dimensions.

There exists no other 2×2 matrix anticommuting with the gamma matrices, which would allow the introduction of a γ^5-matrix in the irreducible representation.

Chiral symmetry

The concept of **chiral symmetries** and their breakdown by mass terms can nevertheless be realized also in the framework of **(2+1)-dimensional quantum field theories**

by considering a **four-component reducible representation for Dirac fields**

The Dirac spinors ψ have the following form:

$$\psi(x) = \begin{pmatrix} \tilde{\psi}_1(x) \\ \tilde{\psi}_2(x) \end{pmatrix},$$

with $\tilde{\psi}_1, \tilde{\psi}_2$ being two-component spinors.
4×4 γ-matrices:

$$\gamma^\mu = diag(\tilde{\gamma}^\mu, -\tilde{\gamma}^\mu)$$

Chiral symmetry: γ^3, γ^5 and τ

There exist two matrices, γ^3 and γ^5, which anticommute with all γ^μ $(\mu = 0, 1, 2)$ and with themselves

$$\gamma^3 = \begin{pmatrix} 0\,, & I \\ I\,, & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0\,, & -I \\ I\,, & 0 \end{pmatrix}$$

One can also construct

$$\tau = -i\gamma^3\gamma^5 = \begin{pmatrix} I\,, & 0 \\ 0\,, & -I \end{pmatrix}$$

Chiral symmetry

The Lagrangian is invariant under two discrete chiral transformations Γ^5 and Γ^3,

$$\Gamma^5 : \qquad \psi_k(t,x,y) \to \gamma^5 \psi_k(t,x,y); \quad \overline{\psi}_k(t,x,y) \to -\overline{\psi}_k(t,x,y)\gamma^5$$

$$\Gamma^3 : \qquad \psi_k(t,x,y) \to \gamma^3 \psi_k(t,x,y); \quad \overline{\psi}_k(t,x,y) \to -\overline{\psi}_k(t,x,y)\gamma^3$$

$\mathcal{P}$ transformation

$$\psi^{\mathcal{P}}(x') \equiv \psi^{\mathcal{P}}(t, -x, y) = \gamma^5\gamma^1\psi(t, x, y)$$

$$\overline{\psi^{\mathcal{P}}}(x') \equiv \overline{\psi^{\mathcal{P}}}(t, -x, y) = \overline{\psi}(t, x, y)\gamma^5\gamma^1$$

$\mathcal{P}$ transformations of some Hermitian bispinor forms such as

$$\begin{aligned}
&\overline{\psi}(x)i\gamma^5\psi(x) \xrightarrow{\mathcal{P}} -\overline{\psi}(x)i\gamma^5\psi(x),\\
&\overline{\psi}(x)i\gamma^3\gamma^5\psi(x) \xrightarrow{\mathcal{P}} -\overline{\psi}(x)i\gamma^3\gamma^5\psi(x),\\
&\overline{\psi}(x)i\gamma^3\psi(x) \xrightarrow{\mathcal{P}} \overline{\psi}(x)i\gamma^3\psi(x).
\end{aligned}$$

$\mathcal{T}$ transformation

$$\psi^{\mathcal{T}}(x') \equiv \psi^{\mathcal{T}}(-t, x, y) = \gamma^5\gamma^2\psi(t, x, y)$$

$$\overline{\psi^{\mathcal{T}}}(x') \equiv \overline{\psi^{\mathcal{T}}}(-t, x, y) = \overline{\psi}(t, x, y)\gamma^5\gamma^2$$

$\mathcal{T}$ transformations of some other Hermitian bispinor forms

$$\overline{\psi}(x)i\gamma^3\psi(x) \xrightarrow{\mathcal{T}} -\overline{\psi}(x)i\gamma^3\psi(x)$$
$$\overline{\psi}(x)i\gamma^5\psi(x) \xrightarrow{\mathcal{T}} -\overline{\psi}(x)i\gamma^5\psi(x)$$
$$\overline{\psi}(x)i\gamma^3\gamma^5\psi(x) \xrightarrow{\mathcal{T}} -\overline{\psi}(x)i\gamma^3\gamma^5\psi(x)$$

Bilocal source for $\overline{\psi}\psi$ composite operators

As a rule one introduces the source terms

$$\mathcal{S}_s = \int d^3x \Big(\overline{\psi}_k(x) J_k(x) + \overline{J}_k(x) \psi_k(y) \Big)$$

But now let us introduce bilocal source

$$\mathcal{S}_s = \int d^3x d^3y \overline{\psi}_k^{\alpha}(x) K_{\alpha}^{\beta}(x,y) \psi_{k\beta}(y)$$

Generating functional

$Z(K)$ is the generating functional of the Green's functions of bilocal fermion-antifermion composite operators $\overline{\psi}_k^\alpha(x)\psi_{k\beta}(y)$

$$Z(K) \equiv \exp(iNW(K)) =$$

$$= \int \mathcal{D}\overline{\psi}_k \mathcal{D}\psi_k \exp\left(i\left[I(\overline{\psi},\psi) + \int d^3x d^3y \overline{\psi}_k^\alpha(x) K_\alpha^\beta(x,y)\psi_{k\beta}(y)\right]\right)$$

where $\alpha, \beta = 1, 2, 3, 4$ are spinor indices,
$K_\alpha^\beta(x,y)$ is a bilocal source of the fermion bilinear composite field $\bar{\psi}_k^\alpha(x)\psi_{k\beta}(y)$

$$I(\overline{\psi},\psi) = \int d^3x d^3y \overline{\psi}_k^\alpha(x) D_\alpha^\beta(x,y)\psi_{k\beta}(y) + I_{int}(\overline{\psi}_k^\alpha \psi_{k\beta})$$

Generating functional

Generating functional can be expressed in the following form

$$Z(K) = \exp\left(iI_{int}\left(-i\frac{\delta}{\delta K}\right)\right)\exp\left[N\mathrm{Tr}\ln\left(D(x,y)+K(x,y)\right)\right]$$

Generating functional

$$Z(K) = \exp(iNW(K))$$

$$\exp(iNW(K)) =$$

$$= \exp\left(iI_{int}\left(-i\frac{\delta}{\delta K}\right)\right)\exp\left[N\mathrm{Tr}\ln\left(D(x,y)+K(x,y)\right)\right]$$

CJT effective action

CJT effective action of the composite bilocal and bispinor operator $\bar{\psi}_k^\alpha(x)\psi_{k\beta}(y)$ is defined as a functional $\Gamma(S)$ of the **full fermion propagator** $S_\beta^\alpha(x,y)$ by **Legendre transformation** of the functional $W(K)$

$$\Gamma(S) = W(K) - \int d^3x d^3y S_\beta^\alpha(x,y) K_\alpha^\beta(y,x),$$

$$S_\beta^\alpha(x,y) = \frac{\delta W(K)}{\delta K_\alpha^\beta(y,x)}.$$

Full fermion propagator

$S(x,y)$ is the full fermion propagator at $K(x,y)=0$

One can show for CJT effective action $\Gamma(S)$

$$\frac{\delta\Gamma(S)}{\delta S^{\alpha}_{\beta}(x,y)} = -K^{\beta}_{\alpha}(y,x)$$

If bilocal sources $K^{\beta}_{\alpha}(y,x)$ are zero, the full fermion propagator is a solution of

$$\frac{\delta\Gamma(S)}{\delta S^{\alpha}_{\beta}(x,y)} = 0.$$

CJT effective action

we calculate the effective action pertubatively

$$\Gamma(S) = -i\mathrm{Tr}\ln\big(-iS^{-1}\big) + \int d^3x d^3y S^{\alpha}_{\beta}(x,y) D^{\beta}_{\alpha}(y,x)$$

$$+\frac{G}{2}\int d^3x \Big[\mathrm{tr} S(x,x)\Big]^2 - \frac{G}{2N}\int d^3x\ \mathrm{tr}\Big[S(x,x)S(x,x)\Big].$$

Stationary equation for the CJT effective action

The stationary equation for the CJT effective action

$$0 = i\left[S^{-1}\right]^{\beta}_{\alpha}(x,y) + D^{\beta}_{\alpha}(x,y) + G\delta^{\beta}_{\alpha}\delta(x-y)\,\mathrm{tr}S(x,y) - \frac{G}{N}S^{\beta}_{\alpha}(x,y)\delta(x-y).$$

$S(x,y)$ is a translationary invariant operator

$$\overline{(S^{-1})^{\beta}_{\alpha}}(p) - ip_{\nu}(\gamma^{\nu})^{\beta}_{\alpha} = iG\delta^{\beta}_{\alpha}\int\frac{d^3q}{(2\pi)^3}\,\mathrm{tr}\overline{S}(q) - i\frac{G}{N}\int\frac{d^3q}{(2\pi)^3}\overline{S^{\beta}_{\alpha}}(q)$$

Dirac mass term generation

Let us explore, using the CJT approach, the possibility of mass term

$$\overline{S^{-1}} = i(\hat{p} + m_D), \quad \text{i.e.} \quad \overline{S} = -i\frac{\hat{p} + m_D}{p^2 - m_D^2}$$

$\mathcal{P}$ - symmetric

$\mathcal{T}$ - symmetric

Break chiral symmetries Γ^5 and Γ^3

The gap equation

$$\frac{m_D}{G} = m_D\Big(4 - \frac{1}{N}\Big)\frac{1}{(2\pi)^3}\int \frac{d^3p}{p^2 + m_D^2}$$

coupling constant renormalization

UV divergence can be removed from the gap equations if we require the following behavior of the bare coupling constant $G \equiv G(\Lambda)$ vs Λ

$$\frac{1}{G(\Lambda)} = \frac{4N-1}{2N\pi^2}\Big(\Lambda + g_D\frac{\pi}{2} + g_D\mathcal{O}\Big(\frac{g_D}{\Lambda}\Big)\Big)$$

where g_D is a finite Λ-independent and renormalization group invariant quantity, and it can also be considered as a new free parameter of the model.

Dynamical mass generation

$$m_D \left(g_D + |m_D| \right) = 0$$

- at $g_D > 0$ its global minimum lies at the point $m_D = 0$, and no dynamical mass generation
- at $g_D < 0$ the global minimum is achieved at $m_D = |g_D|$

$$m_D = |g_D|$$

Renormalization group equation

One could define dimensionless bare coupling constant

$$\lambda = \Lambda G(\Lambda)$$

The β-function is

$$\beta(\Lambda) = \Lambda \frac{\partial \lambda(\Lambda)}{\partial \Lambda}, \qquad \beta(\Lambda) = \frac{\lambda}{\lambda_D}(\lambda_D - \lambda)$$

where $\lambda_D = \frac{2N\pi^2}{4N-1}$

there exists a nonzero UV-stable fixed point λ_D in the model

Renormalization group equation

At rather large values of Λ

$$\lambda(\Lambda) - \lambda_D \sim -\frac{g_D}{\Lambda}$$

- at $\lambda > \lambda_D$ — **chiral symmetry is broken**
- at $\lambda < \lambda_D$ — **symmetry of the model remains intact**

Haldane mass term generation

Let us explore, using the CJT approach, the possibility of mass term

$$\overline{S^{-1}} = i(\hat{p} + \tau m_H), \quad \text{i.e.} \quad \overline{S} = -i\frac{\hat{p} + \tau m_H}{p^2 - m_H^2}$$

$\mathcal{P}$ **- breaking** $\mathcal{T}$ **- symmetric**

Keep chiral symmetries Γ^5 and Γ^3 intact

Coupling constant renormalization

the UV divergence can be removed from the gap equations if we require the following behavior of the bare coupling constant $G \equiv G(\Lambda)$ vs Λ

$$\frac{1}{G(\Lambda)} = -\frac{1}{2N\pi^2}\left(\Lambda + g_H\frac{\pi}{2} + g_H\mathcal{O}\left(\frac{g_H}{\Lambda}\right)\right)$$

where g_H is a finite Λ-independent and renormalization group invariant quantity, and it can also be considered as a new free parameter of the model.

Dynamical generation of Haldane mass

$$m_H \left(g_H + |m_H| \right) = 0$$

- at $g_H > 0$ its global minimum lies at the point $m_H = 0$, and no dynamical generation of Haldane mass
- at $g_H < 0$ the global minimum is achieved at $m_H = |g_H|$

$$m_H = |g_H|$$

Renormalization group equation

At rather large values of Λ

$$\lambda(\Lambda) - \lambda_H \sim \frac{2\pi^2 N g_H}{\Lambda}$$

where $\lambda_H = -2N\pi^2$

- at $\lambda > \lambda_H$ — **parity remains intact**
- at $\lambda < \lambda_H$ — **parity is broken**

$\lambda < \lambda_H$ — **parity is broken**

Since $\lambda_H \to -\infty$ at $N \to \infty$

we may conclude that **in the limit of large** N the (2+1)-D GN model cannot have a **P-odd phase** and **Haldane mass cannot arise dynamically**

More exotic mass terms

Let us explore the possibility that the solution of the gap equation has the form

$$\overline{S^{-1}} = i(\hat{p} + i\gamma^5 m_5 + i\gamma^3 m_3), \quad \text{i.e.} \quad \overline{S} = -i\frac{\hat{p} + i\gamma^5 m_5 + i\gamma^3 m_3}{p^2 - (m_3^2 + m_5^2)}$$

It corresponds to a dynamically generated mass term of the form $\mathcal{M}_H = \left(m_5 \overline{\psi} i\gamma^5 \psi + m_3 \overline{\psi} i\gamma^3 \psi\right)$ in the Lagrangian

Since m_5 and m_3 are some real numbers, this mass term is a Hermitian one.

the UV divergence can be removed from the gap equations if we require the following behavior of the bare coupling constant $G \equiv G(\Lambda)$ vs Λ

$$\frac{1}{G(\Lambda)} = \frac{1}{2N\pi^2}\left(\Lambda + g\frac{\pi}{2} + g\mathcal{O}\left(\frac{g}{\Lambda}\right)\right)$$

where g is a finite Λ-independent and renormalization group invariant quantity, and it can also be considered as a new free parameter of the model.

Dynamical generation of masses m_3 and m_5

- at $g > 0$ only a trivial solution of the gap equations exists, $m_3 = m_5 = 0$, and all discrete symmetries of the model remain intact

- at $g < 0$

$$m_3 = |g| \cos\alpha, m_5 = |g| \sin\alpha$$

(where $0 \leq \alpha \leq \pi/2$ is some arbitrary fixed angle)

Vacuum solution

At $g < 0$ the system undergoes dynamic generation of the masses

$$m_3 = |g| \cos\alpha, \qquad m_5 = |g| \sin\alpha$$

if $\alpha \neq 0, \pi/2$ violation of all discrete symmetries is realized in the model.

if $\alpha = 0$ then only -odd $m_3 = |g|$ mass term is generated, and Γ^3 chiral symmetry is dynamically violated.

if $\alpha = \pi/2$ only -even $m_5 = |g|$ mass term appears dynamically, and in this case chiral Γ^5 symmetry is broken spontaneously.

Physical fermion mass

At $g < 0$ in all above mentioned cases (at arbitrary values of the angle parameter α)
the genuine physical fermion mass, which is indeed a pole of the fermion propagator, is equal to

$$M_F = \sqrt{m_3^2 + m_5^2} \equiv |g|$$

Renormalization group equation

At rather large values of Λ

$$\lambda(\Lambda) - \lambda_{35} \sim \frac{2\pi^2 N g}{\Lambda}$$

- at $\lambda > \lambda_{35}$ — $m_5 \overline{\psi} i \gamma^5 \psi + m_3 \overline{\psi} i \gamma^3 \psi$ **mass term is dynamically generated**
- at $\lambda < \lambda_{35}$ — **symmetric phase**

at $\lambda > \lambda_{35}$ — $m_5\overline{\psi}i\gamma^5\psi + m_3\overline{\psi}i\gamma^3\psi$ **mass term is dynamically generated**

where $\lambda_{35} = 2N\pi^2$

Since $\lambda_{35} \to \infty$ at $N \to \infty$

we may conclude that **in the limit of large** N there is no **dynamical** $m_5\overline{\psi}i\gamma^5\psi + m_3\overline{\psi}i\gamma^3\psi$ **mass term generation**

More exotic masses: non-Hermitian mass terms

$$\mathcal{M}_H = im_5\overline{\psi}(x)\gamma^5\psi(x) + im_3\overline{\psi}(x)\gamma^3\psi(x)$$

$$\mathcal{M}_{NH1} = im_5\overline{\psi}(x)\gamma^5\psi(x) + m_3\overline{\psi}(x)\gamma^3\psi(x)$$

$\mathcal{PT}$ - symmetric

$$\mathcal{M}_{NH2} = m_5\overline{\psi}(x)\gamma^5\psi(x) + im_3\overline{\psi}(x)\gamma^3\psi(x)$$

$\mathcal{PT}$ - breaking

non-Hermitian mass term

Let us explore, using the CJT approach, the possibility of the dynamic appearance of a non-Hermitian and $\mathcal{PT}$ symmetric mass term $\mathcal{M}_{NH1}$

$$\overline{S^{-1}} = i(\hat{p} + i\gamma^5 m_5 + \gamma^3 m_3), \quad \text{i.e.} \quad \overline{S} = -i\frac{\hat{p} + i\gamma^5 m_5 + \gamma^3 m_3}{p^2 - (m_5^2 - m_3^2)}$$

where m_3 and m_5 are real quantities.

Suppose that $m_5^2 \geq m_3^2$

Vacuum solution

- at $g > 0$ its global minimum lies at the point $m_5 = m_3 = 0$, and dynamical mass generation is absent
- at $g < 0$ the global minimum is achieved at arbitrary (m_3, m_5) point such that $m_5^2 - m_3^2 = g^2$

$$m_3 = |g| \sinh \beta, \quad m_5 = |g| \cosh \beta$$

Note that such a structure of the global minimum point of the model appears due to the emergent symmetry of the CJT effective potential with respect to non-Unitary transformations

$$\begin{pmatrix} m_5 \\ m_3 \end{pmatrix} \to \begin{pmatrix} \cosh \beta & \sinh \beta \\ \sinh \beta & \cosh \beta \end{pmatrix} \begin{pmatrix} m_5 \\ m_3 \end{pmatrix}.$$

non-Hermitian mass term

the non-Hermitian but -odd mass term $\mathcal{M}_{NH2}$

$$\overline{S^{-1}} = i(\hat{p} + \gamma^5 m_5 + i\gamma^3 m_3), \quad \text{i.e.} \quad \overline{S} = -i\frac{\hat{p} + \gamma^5 m_5 + i\gamma^3 m_3}{p^2 - (m_3^2 - m_5^2)}$$

where m_3 and m_5 are real quantities.

Suppose that $m_5^2 \geq m_3^2$

non-Hermitian mass term

It can be shown in exactly the same way that for the same dependence of the bare coupling constant G vs Λ, there exists a nontrivial solution of the renormalized stationary (Dyson-Schwinger) equation

- at $g < 0$ of the non-Hermitian but $\mathcal{PT}$-odd mass term $\mathcal{M}_{NH2}$ in the model.

$$m_3 = |g| \cosh \omega, \quad m_5 = |g| \sinh \omega$$

$\sqrt{m_3^2 - m_5^2} \equiv |g|$

fermion pole mass $M_F = \sqrt{m_3^2 - m_5^2} \equiv |g|$.

Renormalization group equation

At rather large values of Λ

$$\lambda(\Lambda) - \lambda_{35} \sim -\frac{2\pi^2 N g}{\Lambda}$$

- at $\lambda > \lambda_{35}$ — **non-Hermitian mass term could be dynamically generated**
- at $\lambda < \lambda_{35}$ — **no non-Hermitian mass terms is dynamically generated**

at $\lambda > \lambda_{35}$ — **non-Hermitian mass terms could be dynamically generated**

where $\lambda_{35} = 2N\pi^2$

Since $\lambda_{35} \to \infty$ at $N \to \infty$

we may conclude that **in the limit of large** N there is no **dynamical generation of non-Hermitian mass terms**

Conclusions and Thanks for the Attention

- There has been studied the possibility of the dynamical appearance of both Hermitian and non-Hermitian mass terms in the originally Hermitian massless (2+1)-dimensional GN model
- the effect of spontaneous non-Hermiticity can be detected only outside the large-N expansion technique
- There has been shown that parity breaking Haldane mass can be generated dynamically in the model