Quadratic Gravity and Non-Conservativity of Energy-Momentum Tensor

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Quadratic Gravity

Lagrangian

- Standard

$$\mathcal{L}_2 = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda$$

$R^{\mu}{}_{\nu\lambda\sigma}$ — Riemann curvature tensor
$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ — Ricci tensor
$R^{\lambda}_{\lambda} = R$ — curvature scalar

- Rearranged

$$\mathcal{L}_2 = \alpha C^2 + \beta GB + \gamma R^2 + \alpha_4 R + \alpha_5 \Lambda$$

$C^2 = C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}, \quad C_{\mu\nu\lambda\sigma}$ — Weyl tensor

$$\begin{aligned} C_{\mu\nu\lambda\sigma} &= R_{\mu\nu\lambda\sigma} + \frac{1}{2}(R_{\mu\sigma} g_{\nu\lambda} + R_{\nu\lambda} g_{\mu\sigma} - R_{\mu\lambda} g_{\nu\sigma} - R_{\nu\sigma} g_{\mu\lambda}) \\ &\quad + \frac{1}{6} R (g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\sigma} g_{\nu\lambda}) \\ C^2 &= R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \end{aligned}$$

GB — Gauss–Bonnet term

$$GB = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$$\begin{cases} \alpha + \beta = \alpha_1 \\ -2\alpha - 4\beta = \alpha_2 \\ \frac{1}{3}\alpha + \beta + \gamma = \alpha_3 \end{cases} \quad \Bigg| \quad \begin{cases} \alpha = 2\alpha_1 + \frac{1}{2}\alpha_2 \\ \beta = \alpha_2 - \frac{1}{2}\alpha_2 \\ \gamma = \alpha_3 + \frac{1}{3}(\alpha_1 + \alpha_2) \end{cases}$$

Total action

$$S_{\rm tot} = S_2 + S_{\rm matter}, \quad S_2 = \int \mathcal{L}_2\sqrt{-g}\,d^4x, \quad S_{\rm matter} = \int \mathcal{L}_{\rm m}\sqrt{-g}\,d^4x$$

Energy-momentum tensor $T^{\mu\nu}$

$$\delta S_{\rm matter} \stackrel{def}{=} -\frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}d^4x = \frac{1}{2}\int T_{\mu\nu}\delta g^{\mu\nu}d^4x$$

Field equations

$$\alpha_4 G_{\mu\nu} - \frac{\alpha_5}{2}\Lambda = -\frac{1}{2}T_{\mu\nu}[R])$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ — Einstein tensor

Conservativity:

$$G^{\nu}_{\mu;\nu} = 0$$

- αC^2

$$B_{\mu\nu} = \frac{1}{8\alpha} T_{\mu\nu}[C^2])$$

$B_{\mu\nu}$ — Bach tensor

$$B_{\mu\nu} = C_{\mu\lambda\nu\sigma}{}^{;\sigma;\lambda} + \frac{1}{2} R^{\lambda\sigma} C_{\mu\lambda\nu\sigma}$$

Conservativity:

$$B^{\nu}_{\mu;\nu} = 0$$

- βGB — no contribution
- γR^2

$$\mathcal{D}_{\mu\nu} = \frac{1}{4\gamma} T_{\mu\nu}[R^2])$$

$$\mathcal{D}_{\mu\nu} = R_{;\mu\nu} - (R_{;\lambda;\kappa} g^{\lambda\kappa}) g_{\mu\nu} - R(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu})$$

Conservativity:

$$\mathcal{D}^{\nu}_{\mu;\nu} = (R^{;\nu}_{;\mu;\nu} - R^{;\nu}_{;\nu;\mu}) - R_{;\nu} R^{\nu}_{\mu}$$

$$\mathcal{D}^{\nu}_{\mu;\nu} = 0 \quad (!)$$

- Conservativity in the bulk
- But (!)
- Singular hypersurface (Σ_0)

$$T^{\mu\nu} = S^{\mu\nu}\delta(n) + T^{\mu\nu}(+)\Theta(n) + T^{\mu\nu}(-)\Theta(-n)$$

$n(x^{\mu}) = 0$ — equation for Σ_0
$\delta(n)$ — Dirac's δ-function
$\Theta(n)$ — Heaviside step-function

- Conservativity equation on singular hypersurface Σ_0

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= S^{\mu\nu}\delta'(n)n_{,\nu} + S^{\mu\nu}{}_{;\nu}\delta(n) + T^{\mu\nu}(+)\delta(n)n_{,\nu} + T^{\mu\nu}{}_{;\nu}(+)\Theta(n) \\ &- T^{\mu\nu}(-)\delta(n)n_{,\nu} + T^{\mu\nu}{}_{;\nu}(-)\Theta(-n) \end{aligned}$$

$T^{\mu\nu}{}_{;\nu}(+) = T^{\mu\nu}{}_{;\nu}(-) = 0 \quad \Longrightarrow$

$$T^{\mu\nu}{}_{;\nu} = S^{\mu\nu}\delta'(n)n_{,\nu} + S^{\mu\nu}{}_{;\nu}\delta(n) + [T^{\mu\nu}]\delta(n)n_{,\nu}$$

- Gauss normal coordinate system

$$ds^2 = \epsilon dn^2 + \gamma_{ij}dx^i dx^j$$

$$T^{\mu\nu}{}_{;\nu} = S^{\mu\nu}\delta'(n) + (S^{\mu\nu}{}_{;\nu} + [T^{\mu\nu}])\delta(n)$$

- Integration: $f(n, x^i)$ — arbitrary function with compact support

$$-(fS^{\mu\nu})_{,n} + f(S^{\mu\nu}{}_{;\nu} + [T^{\mu\nu}]) = fc^{\mu} \quad \Longrightarrow$$

$$b(x^i)S^{\mu n} - S^{\mu n}{}_{,n} - S^{\mu\nu}{}_{;\nu} + [T^{\mu\nu}] = c^{\mu}$$

$$b(x^i) = -\frac{f_{,n}}{f}(n=0) = b^{(\mu)}$$

$$\begin{cases} (b^{(n)} - K)S^{nn} + S^{np}_{\ \ |p} + \epsilon K_{lp} S^{lp} [T^{nn}] = c^n \\ (b^{(i)} - K)S^{in} - K^i_l S^{nl} + S^{ip}_{\ \ |p} + [T^{in}] = c^i \end{cases}$$

K_{lp} — extrinsic curvature of Σ_0, $K_{lp} = -\frac{1}{2}\gamma^{ij}_{,n}$

- General Relativity Einstein Equations on Σ_0=Israel Equations

$$\epsilon([K_{ij}] - g_{ij}[K]) = 8\pi G S_{ij}$$

plus

$$S^{nn} = 0, \quad S^{ni} = 0$$

$K_{lp}\big|_{\Sigma_0}$ — ?

Example: "Vacuum burning" phenomenon
(V.A.Berezin, V.F.Kuzmin, I.I.Tkachev 1983, 1984)

- Spherically symmetric bubble with $S^0_0 = 0$

Only surface tension $S^2_2 = S^3_3$

$$S^0_0 = 0 \quad \Longrightarrow \quad [K^2_2] = 0, \quad K_{lp} S^{lp} = 2K^2_2 S^2_2 \quad \to \quad O.K.$$

Quadratic Gravity

$$
\begin{aligned}
\mathcal{L}_2 &= \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda \\
&= \alpha C^2 + \beta GB + \gamma R^2 + \alpha_4 R + \alpha_5 \Lambda
\end{aligned}
$$

$$T^{\mu\nu} = S^{\mu\nu}\delta(n) + T^{\mu\nu}(+)\Theta(n) + T^{\mu\nu}(-)\Theta(-n)$$

Unlike General Relativity:

- Field Equations

GR — 2nd order in derivatives of metric tensor $g_{\mu\nu}$
QG — 4th order

- Singular hypersurface Σ_0

GR — $S^{\mu\nu} \neq 0 \quad \rightarrow \quad \delta$-function in curvature

$$[K_{ij}] \neq 0$$

QG — δ^2 in Lagrangian (generic case) forbidden

Lichnerowicz conditions

$$[g_{\mu\nu,\lambda}] = 0$$

$$([\Gamma^{\lambda}_{\mu\nu}] = 0 \quad \Longrightarrow \qquad [K_{ij}] = 0)$$

At most

$$[R^{\mu}{}_{\nu\lambda\sigma}] \neq 0 \quad \Longrightarrow \quad [K_{lp,n}] \neq 0$$

In the field equations we obtain both

δ-function = "thin shell" and
δ'-function = "double layer"

- Structure of the field equations on Σ_0:

$$\{ijpl\}[K_{pl,n}](\delta K_{ij}) + \{lp\}[K_{lp,n}](\delta g_{nn}) + \{ijpl\}[K_{lp,n|j}](\delta g_{in})$$
$$+\{ij\}(\delta g_{ij}) = \frac{1}{2} S^{\mu\nu}(\delta g_{\mu\nu})$$

Look !

• There is δK_{ij} in l.h.s. and no such structure in r.h.s.

• δK_{ij} is not an independent variation.

It depends on our choice of solutions in $(\pm)$-bulk regions as well as $\delta\gamma_{ij}$ are

• But, there exists functional freedom.
Thus,

$$(\delta K_{ij}) = B_{ij}^{i'j'}(\delta\gamma_{ij})$$

($\delta g_{ij} = \delta\gamma_{ij}$ due to the use of Gauss normal coordinate system)

• The appearance of arbitrary tensor $B_{ij}^{i'j'}$ is the allusion to the existence of δ'-function in the field equations

• Given the solutions in the bulk and the singular hypersurface Σ_0 the $\{ij\}$-equations serve to determine B

Double layer equations

$$\begin{cases} \epsilon\left\{\alpha K^{lp} + (2\gamma - \frac{1}{3}\alpha) K g^{lp}\right\} [K_{lp,n}] = \frac{1}{4} S^{nn} \\ \left\{\alpha g^{il} g^{ip} + (2\gamma - \frac{1}{3}\alpha) g^{ij} g^{lp}\right\} [K_{lp,n|j}] = \frac{1}{4} S^{ni} \\ [K_{lp}] = 0 \end{cases}$$

The necessity of appearance of S^{nn} and S^{ni} was noticed and emphasized by J.M.M.Senovilla (2015)

• The nonzero values S^{nn} and S^{ni} indicate directly to the possibility of matter creation by the double layers

Example:

- Conformal gravity: $\mathcal{L}_2 = \alpha C^2$
- Spherical symmetry
- Bubble with empty space-time inside.
  Double layer as a boundary
- No mater fields — pure gravitation ($S^{ij} = 0$).
  Vacuum solution outside
- Search for the solution for double layer — no such solution.
  Double layer is collapsing
- When collapsing, it radiates
- In the case of absent of the double layer, i. e., with only thin shell ($S^0_0 \neq 0$, $S^0_0 + 2S^2_2 = 0$), the result is $S^0_0 = const$.
  And no creation of matter!

Thanks to all

The End