Nonstationary configurations of a massless scalar field

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Action and stress-energy tensor

We study nonstationary spherically symmetric solutions of the Einstein-scalar field system with a massless scalar field minimally coupled to gravity. We begin with the action

\[ S = \int \left( -\frac{1}{2} S + \langle d\phi, d\phi \rangle \right) \sqrt{|g|} d^4x. \]

The components of the stress-energy tensor are determined by formula

\[ T_{ij} = 2 \partial_i \phi \partial_j \phi - (g^{km} \partial_k \phi \partial_m \phi) g_{ij}. \]
Einstein and Klein-Gordon equations

\[ R_{ij} - \frac{1}{2} S g_{ij} = T_{ij}, \quad \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j \phi) = 0, \quad g = \det(g_{ij}), \]

A configuration will be called stationary if \( \phi = \phi (C) \), where \( C \) is the radius of the sphere. Otherwise, the configuration will be called nonstationary.

This allows us to use the coordinate system

\[(\phi, C, \theta, \varphi),\]

at least locally, for any nonstationary configuration.
Method for constructing nonstationary configurations of a spherically symmetric scalar field

Characteristic function:

\[ f = -\langle dC, dC \rangle. \]

Metric in coordinates \((\phi, C, \theta, \varphi)\):

\[
    ds^2 = -4 \frac{C^2 f d\phi^2 + Cf'_\phi dC d\phi + (Cf'_C + f - 1) dC^2}{4f(Cf'_C + f - 1) - (f'_\phi)^2} - C^2(d\theta^2 + \sin^2 \theta d\varphi^2).
\]

The behavior of function \(f(\phi, C)\) makes it possible to interpret the solution as a black hole, wormhole, or naked singularity.
Method for constructing nonstationary configurations of a spherically symmetric scalar field

The equation for the characteristic function is equivalent to the Klein-Gordon equation, which takes the form

\[
\frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j \phi \right) = 0 \iff
\]

\[
(1 - f - f_C'C)\phi_{\phi\phi}'' - f''_{cc}C^2f + Cf_{\phi c}f_{\phi}'' + (f_C')^2C^2 + (3Cf - 3C)f_C' +
+ 4f^2 - 6f + 2 = 0. \quad (*)
\]

Coordinate system \((t, C, \theta, \varphi)\):

\[
ds^2 = -\frac{4C^2fdt^2}{\left(4f(Cf_C' + f - 1) - (f_\phi')^2\right)\left(t_\phi'\right)^2} - \frac{dC^2}{f} - C^2(d\theta^2 + \sin^2\theta d\varphi^2),
\]

\[
\frac{t'_C}{t'_\phi} = \frac{f_\phi'}{2Cf}
\]
Classes of metric functions

I. \( f = f(\phi) \).

Form of the equation

\[
\frac{d^2}{d\phi^2} f(\phi) - 4f(\phi) + 2 = 0 \implies f(\phi) = \frac{1}{2} + C_1 e^{2\phi} + C_2 e^{-2\phi}.
\]

Metric in coordinates \((\phi, C, \theta, \varphi)\):

\[
ds^2 = 4 \left( \frac{C^2 \left( \frac{1}{2} + C_1 e^{2\phi} + C_2 e^{-2\phi} \right) d\phi^2 + 2C(C_1 e^{2\phi} - C_2 e^{-2\phi}) dC d\phi + \left( -\frac{1}{2} + C_1 e^{2\phi} + C_2 e^{-2\phi} \right) dC^2}{1 - 16C_1 C_2} \right. \\
\quad \quad - C^2 (d\theta^2 + \sin^2\theta d\varphi^2),
\]

The metric signature \((+ - - -)\) entails the condition

\[
1 - 16C_1 C_2 > 0.
\]
Next, we revert to the usual coordinates \((t,C,\theta,\phi)\)

\[
t = Ce^{\int \frac{1+C_1 e^{2\phi}+C_2 e^{-2\phi}}{C_1 e^{2\phi} - C_2 e^{-2\phi}} d\phi}.
\]

Schwarzschild asymptotics are possible if the following conditions are met:

\[
f(\phi) = \frac{1}{2} + C_1 e^{2\phi} + C_2 e^{-2\phi} = 1,
C_1 e^{2\phi} - C_2 e^{-2\phi} = 0 \implies 1 - 16C_1 C_2 = 0 - \text{degenerate case.}
\]
Classes of metric functions

II. \( f (\phi, C) = 1 + C^2 h(\phi). \)

Klein-Gordon equation

\[
3h''(\phi)h(\phi) - 2(h'(\phi))^2 - 12(h(\phi))^2 = 0 \implies h(\phi) = \left(C_1 e^{\frac{2\sqrt{3}}{3} \phi} + C_2 e^{-\frac{2\sqrt{3}}{3} \phi}\right)^3.
\]

The exact form of the metric in the coordinates \((\phi, C, \theta, \varphi)\)

\[
ds^2 = -\frac{4}{\Delta} \left((1 + C^2 h(\phi)) d\phi^2 + Ch'(\phi)d\phi dC + 3h(\phi) dC^2\right) - C^2 d\Omega^2,
\]

\[
\Delta = 12h(\phi)(1 + C^2 h(\phi)) - C^2 (h'(\phi))^2 < 0.
\]
Coordinates \((t,C,\theta,\phi)\):

\[
t = \frac{2C_1 C^2 \left( C_1^2 e^{\frac{8\sqrt{3}}{3} \phi} - C_2^2 \right) - e^{\frac{2\sqrt{3}}{3} \phi}}{C_1 e^{\frac{2\sqrt{3}}{3} \phi} \left( C_1 e^{\frac{4\sqrt{3}}{3} \phi} + C_2 \right)}.
\]

\[C_1 = 1, \quad C_2 = -1, \quad f = 1 + 8C^2 \sinh^3 \frac{2\sqrt{3}}{3} \phi,\]

\[
\phi > 0, \quad C^2 > \frac{1}{8 \sinh \frac{2\sqrt{3}}{3} \phi}
\]

\[
\phi \sim \frac{\sqrt{3}}{16C^2}, \quad C \to \infty, \quad t = \text{Const}, \quad t > \frac{1}{2}
\]

\[
\phi < 0
\]

\[
t > 1, \quad f = 0 \Leftrightarrow \phi = -\frac{\sqrt{3}}{2} \arsh \left( \frac{1}{2C^3} \right)
\]
Metric in coordinates \((t, C, \theta, \varphi)\):

\[
ds^2 = -\frac{4C^2 f dt^2}{\left(4f(Cf_C' + f - 1) - (f')^2\right)(t')^2} - \frac{dC^2}{f} - C^2(d\theta^2 + \sin^2\theta d\varphi^2).
\]

\[\phi > 0, C^2 > \frac{1}{8\text{sh} \frac{2\sqrt{3}}{3} \phi}\]

\[\phi < 0\]
Classes of metric functions

III. $f = f(C)$.

Scalar field equation

$-f''_{cc} C^2 f + (f'_c)^2 C^2 + 3C(f - 1)f'_c + 4f^2 - 6f + 2 = 0.$

Under the assumption that the scalar field depends on time only ($\phi = t$), the metric can be written as

$$ds^2 = \frac{C^2 dt^2}{1 - C f'_c - f} - \frac{dC^2}{f} - C^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

$$(1 - C f'_c - f)f > 0.$$
Direct substitution of the series for the characteristic function into the Klein-Gordon equation allows us to conclude that there are no solutions with Schwarshild asymptotics in this case.

Also there are no solutions with de Sitter asymptotics.

**Numerical solutions**
Conclusions:

• Scalar field equation in coordinate system \((\phi, C, \theta, \varphi)\)

\[
(1 - f - f'_C C)f''_{\phi\phi} - f''_{CC} C^2 f + C f_{\phi C} f'_\phi + (f'_C)^2 C^2 + (3Cf - 3C)f'_C + 4f^2 - 6f + 2 = 0
\]

allows one to obtain both exact and numerical solutions for a massless scalar field.

• Exact solutions are obtained for a massless scalar field. These solutions are related to characteristic functions of a special kind. Analysis of specific exact solutions can help clarify the general features of nonstationary scalar field configurations.

• Studying the behavior of the characteristic function contributes to a more correct formulation of the problem of obtaining numerical nonstationary solutions.
Thank you for attention!