Nonstationary configurations of a massless scalar field

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Action and stress-energy tensor

We study nonstationary spherically symmetric solutions of the Einstein-scalar field system with a massless scalar field minimally coupled to gravity.

We begin with the action

$$S = \int \left(-\frac{1}{2} S + \langle d\phi, d\phi \rangle \right) \sqrt{|g|} d^4x \,.$$

The components of the stress-energy tensor are determined by formula

$$T_{ij} = 2\partial_i \phi \partial_j \phi - (g^{km} \partial_k \phi \partial_m \phi) g_{ij} \,.$$

Einstein and Klein-Gordon equations

$$R_{ij} - \frac{1}{2} S g_{ij} = T_{ij} \,, \quad \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \phi \right) = 0, \quad g = \det\left(g_{ij}\right),$$

A configuration will be called stationary if $\phi = \phi\,(C)$, where C is the radius of the sphere. Otherwise, the configuration will be called nonstationary.

This allows us to use the coordinate system

$$(\phi, C, \theta, \varphi),$$

at least locally, for any nonstationary configuration.

Method for constructing nonstationary configurations of a spherically symmetric scalar field

Characteristic function:

$$f = -\langle dC, dC \rangle.$$

Metric in coordinates $(\phi, C, \theta, \varphi)$:

$$ds^2 = -4\frac{C^2 f d\phi^2 + C f'_\phi dC d\phi + (C f'_C + f - 1) dC^2}{4f(C f'_C + f - 1) - \left(f'_\phi\right)^2} - C^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The behavior of function $f(\phi, C)$ makes it possible to interpret the solution as a black hole, wormhole, or naked singularity.

Method for constructing nonstationary configurations of a spherically symmetric scalar field

The equation for the characteristic function is equivalent to the Klein-Gordon equation, which takes the form

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi\right)=0 \iff$$

$$(1-f-f'_C C)f''_{\phi\phi}-f''_{CC}C^2 f+Cf''_{\phi C}f'_\phi+(f'_C)^2C^2+(3Cf-3C)f'_C+$$

$$+4f^2-6f+2 \; =0. \qquad (*)$$

Coordinate system (t, C, θ, φ) :

$$ds^2=-\frac{4C^2 f dt^2}{\left(4f(Cf'_C+f-1)-\left(f'_\phi\right)^2\right)\left(t'_\phi\right)^2}-\frac{dC^2}{f}-C^2(d\theta^2+\sin^2\theta d\varphi^2),$$

$$\frac{t'_C}{t'_\phi}=\frac{f'_\phi}{2Cf}$$

Classes of metric functions

I. $f = f(\phi)$.

Form of the equation

$$f''(\phi) - 4f(\phi) + 2 = 0 \quad \Longrightarrow$$

$$f(\phi) = \frac{1}{2} + C_1\ e^{2\phi} + C_2\ e^{-2\phi}.$$

Metric in coordinates $(\phi, C, \theta, \varphi)$:

$$ds^2 = $$
$$= 4\frac{C^2(\frac{1}{2} + C_1\ e^{2\phi} + C_2\ e^{-2\phi})d\phi^2 + 2C(C_1\ e^{2\phi} - C_2\ e^{-2\phi})dCd\phi + (-\frac{1}{2} + C_1\ e^{2\phi} + C_2\ e^{-2\phi})dC^2}{1 - 16C_1\ C_2}$$
$$- C^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

The metric signature (+ - - -) entails the condition

$$1 - 16C_1\ C_2 > 0\ .$$

Next, we revert to the usual coordinates (t,C,θ,φ)

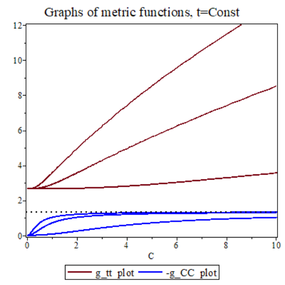
$$t = Ce^{\int \frac{\frac{1}{2}+C_1\ e^{2\phi}+C_2\ e^{-2\phi}}{C_1\ e^{2\phi}-C_2\ e^{-2\phi}} d\phi}.$$

Schwarzschild asymptotics are possible if the following conditions are met:

$$f(\phi) = \frac{1}{2} + C_1\ e^{2\phi} + C_2\ e^{-2\phi} = 1, \quad C_1\ e^{2\phi} - C_2\ e^{-2\phi} = 0 \quad \Longrightarrow$$

$1 - 16C_1\ C_2\ = 0$ - degenerate case.

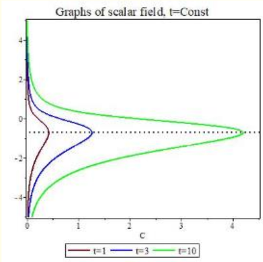
$$C_1 = C_2 = \frac{1}{8}$$



$$t = C\sqrt{th|\phi|}sh\phi,$$

$$|\phi| \sim \left(\frac{t}{C}\right)^{\frac{2}{3}}, C \to \infty$$

$$C_1 = \frac{1}{8}, C_2 = -\frac{1}{8}$$



$$t = C\sqrt{ch(2\phi)}\ e^{2arctg(e^{2\phi})},$$

$$f = 0 \Leftrightarrow \phi = \frac{arsh(-2)}{2}$$

Classes of metric functions

II. $f(\phi, C) = 1 + C^2 h(\phi).$

Klein-Gordon equation

$$3h''(\phi)h(\phi) - 2\big(h'(\phi)\big)^2 - 12\big(h(\phi)\big)^2 = 0 \quad \Longrightarrow$$

$$h(\phi) = \left(C_1\ e^{\frac{2\sqrt{3}}{3}\phi} + C_2\ e^{-\frac{2\sqrt{3}}{3}\phi}\right)^3.$$

The exact form of the metric in the coordinates $(\phi, C, \theta, \varphi)$

$$ds^2 = -\frac{4}{\Delta}\Big(\big(1 + C^2 h(\phi)\big)d\phi^2 + Ch'(\phi)d\phi dC + 3h(\phi)dC^2\Big) - C^2 d\Omega^2,$$

$$\Delta = 12h(\phi)\big(1 + C^2 h(\phi)\big) - C^2\big(h'(\phi)\big)^2 < 0.$$

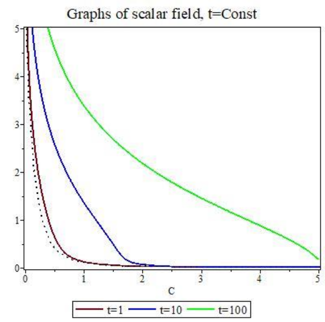
Coordinates (t,C,θ,φ):

$$t = \frac{2C_1\ C^2\left(C_1^2 e^{\frac{8\sqrt{3}}{3}\phi} - C_2^2\right) - e^{\frac{2\sqrt{3}}{3}\phi}}{C_1\ e^{\frac{2\sqrt{3}}{3}\phi}\left(C_1\ e^{\frac{4\sqrt{3}}{3}\phi} + C_2\right)}\ .$$

$$C_1\ = 1,\ \ C_2\ = -1,\ \ f = 1 + 8C^2 sh^3\frac{2\sqrt{3}}{3}\phi,$$

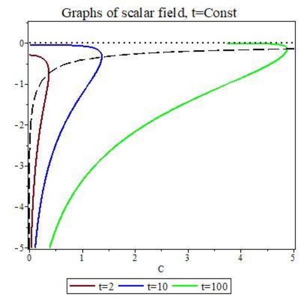
$$\phi > 0,\ C^2 > \frac{1}{8sh\frac{2\sqrt{3}}{3}\phi}$$

$$\phi \sim \frac{\sqrt{3}}{16C^2},\qquad C \to \infty,\qquad t = Const,\qquad t > \frac{1}{2}$$



$$\phi < 0$$

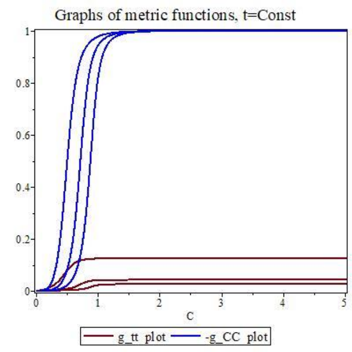
$$t > 1,\qquad f = 0\ \Leftrightarrow\ \phi = -\frac{\sqrt{3}}{2} arsh\left(\frac{1}{2C^{\frac{2}{3}}}\right)$$



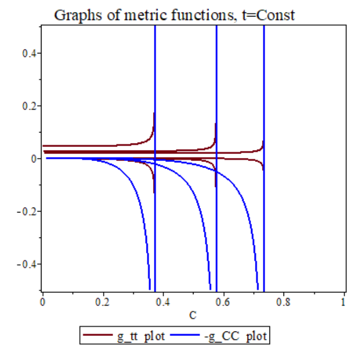
Metric in coordinates (t, C, θ, φ) :

$$ds^2 = -\frac{4C^2 f dt^2}{\left(4f(Cf'_C + f - 1) - \left(f'_\phi\right)^2\right)\left(t'_\phi\right)^2} - \frac{dC^2}{f} - C^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

$\phi > 0,\ C^2 > \dfrac{1}{8sh\frac{2\sqrt{3}}{3}\phi}$



$\phi < 0$



Classes of metric functions

III. $f = f(C)$.

Scalar field equation

$$-f''_{CC}C^2 f + (f'_C)^2 C^2 + 3C(f-1)f'_C + 4f^2 - 6f + 2 = 0.$$

Under the assumption that the scalar field depends on time only ($\phi = t$), the metric can be written as

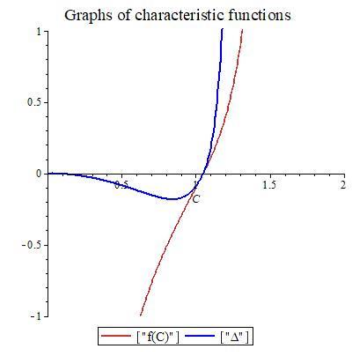
$$ds^2 = \frac{C^2 dt^2}{1 - Cf'_C - f} - \frac{dC^2}{f} - C^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

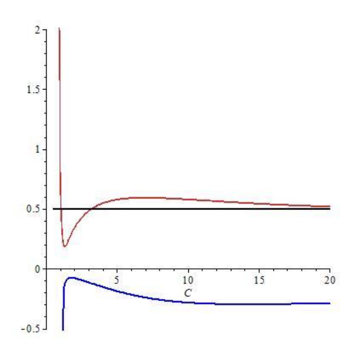
$$(1 - Cf'_C - f)f > 0.$$

Direct substitution of the series for the characteristic function into the Klein-Gordon equation allows us to conclude that there are no solutions with Schwarshild asymptotics in this case.

Also there are no solutions with de Sitter asymptotics.

Numerical solutions





Conclusions:

- Scalar field equation in coordinate system $(\phi, C, \theta, \varphi)$

$$(1 - f - f'_C C) f''_{\phi\phi} - f''_{CC} C^2 f + C f''_{\phi C} f'_\phi + (f'_C)^2 C^2 + (3Cf - 3C) f'_C +$$
$$+4f^2 - 6f + 2 = 0$$

  allows one to obtain both exact and numerical solutions for a massless scalar field.

- Exact solutions are obtained for a massless scalar field. These solutions are related to characteristic functions of a special kind. Analysis of specific exact solutions can help clarify the general features of nonstationary scalar field configurations.

- Studying the behavior of the characteristic function contributes to a more correct formulation of the problem of obtaining numerical nonstationary solutions.

Thank you for attention!