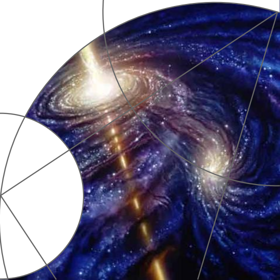
Scattering d-waves on distorted black holes

A.J. Nurmagambetov

ITP KIPT
ajn@kipt.kharkov.ua

also & IRE NASU and KNUK



Here is the Plan:

1. Some Ideology
2. Technical Points
3. Discussion of the results

Ideology

BHs in Modern Astrophysics

The progress in observable astronomy is going so rapid that we become eyewitnesses of transforming astrophysics into an exact experimental discipline.
In particular, one can emphasize the following breakthroughs in the contemporary Multi-messenger Astrophysics, which are related to BHs:

- Registration of Gravitational Waves by LIGO/VIRGO team (Nobel Prize in Physics 2017). It is believed that the GW are mostly induced in processes which involve black holes
- Detection of ultra high energy cosmic rays (UHECRs) of energy $\sim 10^{19}$ eV, far exceeds that of LHC, coming from Active Galactic Nuclei (AGN). It is believed that the SMBH is located in the core of the AGN.
- Revealing the BH Event Horizon (M87 SMBH; $6.5{\times}10^6 M_{\odot}$). It is believed more and more that Astrophysical BHs (BHs in the sky) share common properties with Mathematical BHs (solutions to Einstein equations).
- The recent data analysis of the EHT team puts the Einstein theory on the test (Psaltis et al. PRL 125 2020). It turns out that GR works well.

BHs in Modern Astrophysics

A large enough amount of information on BHs comes with d-waves (spin 2 waves; GWaves). In view of this fact, any studies of d-waves become potentially important. One of the branches in this activity is scattering d-waves on compact objects.

In what follows we will be interested in solving for the scattering problem for the so-called distorted BHs.

Q: What are distorted BHs?

The Israel's uniqueness theorem (Israel 1967) selects the Schwarzschild solution as the only static vacuum BH solution of the Einstein eqs. in an asymtotically flat (AF) spacetime.

For a real astrophysical problem the Schwarzschild solution is highly idealized (even if we do not take into account a possible BH rotation).

Any presence of matter, e.g. in accretion disk, distorts the metric!

BHs in Modern Astrophysics

If a static distribution of matter is localized outside the BH horizon, the spacetime at the vicinity of the horizon remains vacuum spacetime.

The solution of a BH type, describing such a configuration, is called a distorted BH. (Weyl 1917; Geroch&Hartle 1982).

A distortion of the static/stationary BH horizon is the modelling, to some extent, the interaction of a BH with the external matter.

An example of the distorted metric (Schwarzshild in Minkowski ($\kappa = 0$) or AdS ($\kappa \neq 0$); T. Moskalets&AJN 2015, 2016; Boos&Frolov 2018)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 e^{\chi(\theta,\varphi)}(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$f(r) = 1 + \kappa^2 r^2 - \frac{2M}{r}$$

The conformal factor $\chi(\theta,\varphi)$ (the metric potential) obeys the spherical Liouville equation

$$\Delta_{\theta,\varphi}\,\chi(\theta,\varphi) + 2(e^{\chi(\theta,\varphi)} - 1) = 0.$$

$$\Delta_{\theta,\phi} \equiv \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi^2$$

Technical Points

Technical Points

Technically, we would like to extend the known Regge-Wheeler/Zerilli equation for small d-wave perturbations over the known GR background, choosen to be that of a static distorted BH, and try to solve it.

First task on this way is to separate the variables.

Our previous experience with s-wave perturbations over such a background (T. Moskalets&AJN 2016)

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Phi = 0 \quad \rightsquigarrow$$

$$-\frac{1}{f}\partial_t^2\Phi + \frac{1}{r^2}\partial_r(r^2 f\,\partial_r\Phi) + \frac{e^{-\chi(\theta,\varphi)}}{r^2}\triangle_{\theta,\varphi}\,\Phi = 0$$

indicated the separation of variables by choosing the ansatz

$$\Phi(t,r,\theta,\varphi) = e^{-i\omega t}Q_0(r)\Theta(\theta,\varphi)$$

Since the separation is realizable, we are dealing with a special class of the metric (in the Petrov classification).

Technical Points

To determine the (Petrov) type of the metric, let's introduce the Newman-Penrose tetrade $(l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$:

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu$$

For the considered case we explicitly have

$$l_\mu = \delta_{\mu 0} - \frac{\delta_{\mu r}}{f(r)}, \quad n_\mu = \frac{f(r)}{2}\delta_{\mu 0} + \frac{\delta_{\mu r}}{2}, \quad m_\mu = -\frac{e^{\frac{\chi(\theta,\phi)}{2}}}{\sqrt{2}} r(\delta_{\mu\theta} + i \sin\theta \delta_{\mu\varphi})$$

and compute the spin-connection $\gamma_{(c)(a)(b)} = e^\nu_{(c)} e_{(a)\nu;\mu} e^\mu_{(b)}$ with

$$e^\mu_{(1)} = l^\mu, \quad e^\mu_{(2)} = n^\mu, \quad e^\mu_{(3)} = m^\mu, \quad e^\mu_{(4)} = \bar{m}^\mu$$

As a result, the spin-connection coeffs

$$\kappa \equiv \gamma_{(3)(1)(1)}, \qquad \sigma \equiv \gamma_{(3)(1)(3)}$$

$$\nu \equiv \gamma_{(2)(4)(2)}, \qquad \lambda = \gamma_{(2)(4)(4)}$$

vanish that, according to the Goldberg-Sachs theorem, indicates the Petrov-type D metric.

Technical Points

For Type D vacuum background metric equations for gravitational perturbations decouple for

$$\psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \qquad \psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

with the Weyl tensor $C_{\alpha\beta\gamma\delta}$ (S. A. Teukolsky 1973)

Following Newman-Penrose we introduce differential operators, formed by contraction of the NP tetrade with the space-time derivative:

$$D = l^\mu \partial_\mu, \quad \Delta = n^\mu \partial_\mu, \quad \delta = m^\mu \partial_\mu, \quad \bar{\delta} = \bar{m}^\mu \partial_\mu$$

Then the linearized equations for the d-wave are (S. Chandrasekhar The BHs Book 1983)

$$\left[(D - 5\rho)(\Delta - 4\gamma + \mu) - (\delta + 2\bar{\alpha})(\bar{\delta} - 4\alpha) + \frac{3M}{r^3} \right] \psi_0(x) = 0,$$

$$\left[(\Delta + 5\mu + 2\gamma)(D - \rho) - (\bar{\delta} + 2\alpha)(\delta - 4\bar{\alpha}) + \frac{3M}{r^3} \right] \psi_4(x) = 0$$

Technical Points

Eqs. for all type of perturbations (up to d-waves) admit the separation of variables within the ansatz

$$\begin{pmatrix} r^{-1}\Phi \\ \phi_{0,1,2} \\ \psi_0 \\ r^4\psi_4 \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} Q_0(r), \Theta(\theta,\varphi) \\ R_{0,1,2}(r)\,\Theta_{0,1,-1}(\theta,\varphi) \\ \Psi_2(r)\,\Theta_2(\theta,\varphi) \\ \Psi_{-2}(r)\,\Theta_{-2}(\theta,\varphi) \end{pmatrix}$$

The angular part of the separation $\Theta \equiv \Theta_0$ satisfies

$$\Delta_{\theta,\varphi}\Theta(\theta,\varphi) + Ce^{\chi(\theta,\varphi)}\Theta(\theta,\varphi) = 0$$

with the separation constant $C \equiv \nu(\nu+1)$. It's worth mentioning that the spherical symmetry is lost, hence ν (in general) does not fall into integer numbers.

$\Theta_{\pm 1}, \Theta_{\pm 2}$ are formed from $\Theta \equiv \Theta_0$ by use of differential operators

$$L_n = \partial_\theta - \frac{i}{\sin\theta}\partial_\varphi + n\left(\cot\theta + \frac{1}{2}\left(\partial_\theta\chi - \frac{i}{\sin\theta}\partial_\varphi\chi\right)\right)$$

For instance,

$$\Theta_{+1} = e^{-\frac{1}{2}\chi(\theta,\phi)}L_1^\dagger\Theta(\theta,\varphi), \quad \Theta_{-2}(\theta,\varphi) = e^{-\frac{1}{2}\chi(\theta,\phi)}L_{-1}e^{-\frac{1}{2}\chi(\theta,\phi)}L_0\,\Theta(\theta,\varphi)$$

Technical Points

For the radial part, Eqs. for p- and d-waves have the properties of

- Functions $R_{0,2}$ can be obtained from R_1. So, it is enough to consider $Q_1(r) = rR_1$ for p-waves;
- The radial part of d-wave perturbations $\Psi_{\pm 2}$ gets transformed to functions $Q_{\pm 2}$ of the Regge-Wheeler/Zerrilli equations for metric perturbations, directly coming from the linearized Einstein equations (S. Chandrasekhar 1983).

The radial equations comes as follows

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - V_s(r)\right] Q_s = 0, \quad r_* \in (-\infty, +\infty)$$

with r_* "tortoise"coordinate $\frac{dr}{dr_*} = f(r)$ and the effective potential for odd (axial p-, d-wave) perturbations

$$V_s(r) = V_s(r) = \frac{(1-s^2) f \partial_r f}{r} + \nu(\nu+1)\frac{f}{r^2} + 3s(s-1)\kappa^2 f, \quad s = 0, 1, 2;$$

or for even d-waves (with $c = [\nu(\nu+1) - 2]/2$)

$$V_{-2}(r) = \frac{2f(r)}{r^3} \frac{9M^3 + 3c^2 M r^2 + c^2(1+c) r^3 + 9M^2\left(cr + 3\kappa^2 r^3\right)}{(3M + cr)^2}$$

Technical Points

To sum up, for the radial (RW/Zerilli) eqs. we get a 1D Schrodinger-type equation with the effective potential $V_s(r)$ $(V_{-2}(r))$ in which

- the total angular momentum and its projection on a selected axis do not preserve anymore;
- the quantity ν, playing the role of the total angular momentum, takes generally non-integer values;
- if $\chi(\theta, \phi) = 0$ then $\nu \to l \in \mathbb{Z}$ and we get spherically symmetric case back.

Looking at the situation from QM point of view, in presence of the distortion the total angular momentum spectrum is quantized, but not in integers.

Technical Points

Other non-trivialitites come from the angular part of the separation ansatz. The angular equation

$$\Delta_{\theta,\varphi}\Theta(\theta,\varphi) + Ce^{\chi(\theta,\varphi)}\Theta(\theta,\varphi) = 0, \quad C = \nu(\nu+1)$$

is solved (numerically) by use of expansion in spherical harmonics

$$\Theta(\theta,\varphi) = \sum_{j,m}^{\infty} c_{jm} Y_{jm}(\theta,\varphi)$$

Then the equation for $\Theta(\theta,\varphi)$ turns into a generalized eigenvalue problem

$$\sum_{j,m}^{\infty} A_{km',\,jm} c_{jm} = C \sum_{j,m}^{\infty} B_{km',\,jm} c_{jm}, \quad \begin{aligned} A_{km',jm} &= j(j+1)\delta_{kj}\delta_{m'm}, \\ B_{km',jm} &= \int d\Omega e^{\chi(\theta,\varphi)} Y^*_{km'}(\theta,\varphi) Y_{jm}(\theta,\varphi) \end{aligned}$$

for infinite dimensional matrices A and B.

In numerical computations the summation is upper bounded to some j_{max}, at which the eigenvalues and the corresponding eigenvectors do not visibly changed upon j_{max} increasing.

Technical Points

Analysis simplifies for the axially symmetric case, when $\chi = \chi(\theta)$. It turns out that (A. Arslanaliev&AJN, in preparation)

$$e^{\chi(x)} = \frac{(2ab)^2 \left(\frac{1-x}{1+x}\right)^a}{(1-x^2)\left(b^2 + \left(\frac{1-x}{1+x}\right)^a\right)^2}, \quad x = \cos\theta$$

and $a = b = 1$ gives $e^{\chi(x)} = 1$, i.e. spherically symmetric case.

Further separation in angles gives

$$\Theta \to \Theta_m(\theta, \varphi) = e^{im\varphi} S_m(\theta), \quad \Theta_m(\theta, \varphi) = \Theta_m(\theta, \varphi + 2\pi) \to m \in Z$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dS_m(x)}{dx}\right] + \left[\nu(\nu+1)e^{\chi(x)} - \frac{m^2}{1-x^2}\right] S_m(x) = 0, \quad x = \cos\theta$$

and reduces the generalized eigenvalue problem to matrices

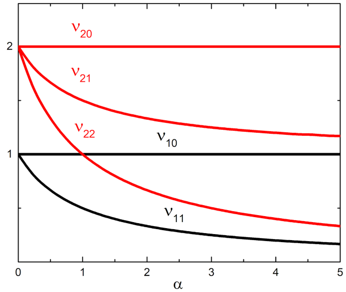
$$A_{ij} = j(j+1)\frac{2(j+m)!}{(2j+1)(j-m)!}\delta_{ij}, \; B_{ij} = \int_{-1}^{1} dx\, e^{\chi(x)} P_{im}(x) P_{jm}(x).$$

Technical Points

The eigenvalues ν's, coming from solving for the generalized eigenvalue problem

- labeled with two indices $\{l, m\}$, $l > 0, m = -l, ...l$, with the triviality condition $\nu_{lm} \to l \in \mathbb{Z}$ once $\chi \to 0$;
- numerics give $\nu_{l0} = l$ and $\nu_{l,-m} = \nu_{lm}$

For example, for $b = 1$ and $a = 1 + \alpha$, we put on the plot first eigenvalues for $m = 1, 2$ as functions of the parameter α. For a positive α, $\nu_{lm} \leq l$



For values $\alpha = [0.2, 0.5]$ numerics give
$\nu_{11} = [0.833, 0.666]$,
$\nu_{21} = [1.833, 1.666]$,
$\nu_{22} = [1.666, 1.333]$,
$\nu_{32} = [2.666, 2.333]$

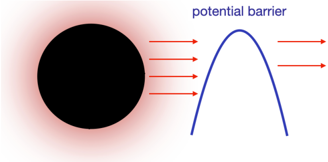
It turns out that for fixed m

$$\nu_{lm} = \nu_{l-1m} + 1$$

Technical Points

Finally, consider the greybody factor (GBF). The GBF $\gamma(\omega)$ is the probability for an outgoing waves of ω frequency to reach infinity. The GBF is an important characterics of the scattering, since it enters the absorption cross-section as well as the spectrum of the Hawking radiation

$$\langle n(\omega) \rangle = \frac{\gamma(\omega)}{e^{\frac{\omega}{T_H}} \pm 1}$$



$\langle n(\omega) \rangle$ - number of the corresponding type particles.

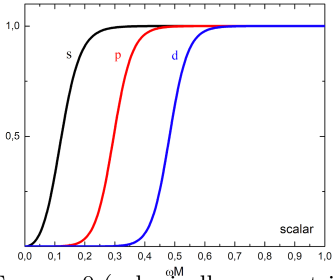
Therefore, to determine the GBF we got 1D scattering problem with boundary conditions (T. Harmark et. al. 2010)

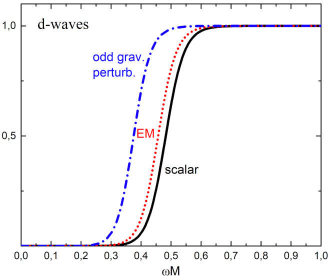
only ingoing waves at the horizon $\quad Q_s(r) = T(\omega) e^{-i\omega r_*}, \ r_* \to -\infty$

ingoing + outgoing waves at spatial infinity $\quad Q_s(r) = e^{-i\omega r_*} + R(\omega) e^{i\omega r_*}, \ \ r_* \to \infty$

$$\gamma(\omega) = |T(\omega)|^2$$

Technical Points



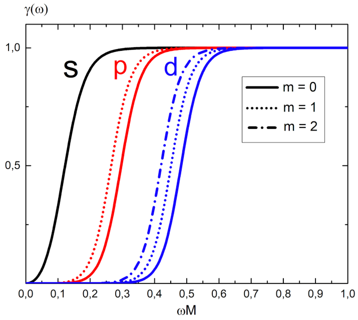


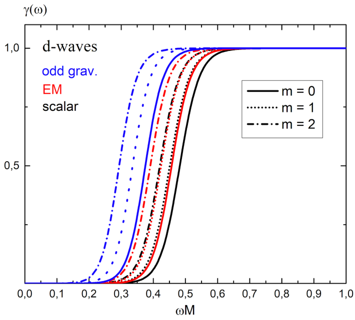
For $\alpha = 0$ (spherically symmetric case)

- scalar s-wave (left panel) has the complete transmission (c.t.) at the lowest value of ω
- increasing l in the scalar mode (p-, d-modes; left panel) requires higher values of ω for the c.t.
- for d-waves ($l = 2$) gravitational perturbations has the c.t. at the lowest frequencies (in compare to $l = 2$ modes of EM waves and scalar waves; right panel).

$\gamma(\omega)$s are computed in the path-ordered-exponential approach (F. Gray, M. Visser, arXiv:1512.05018)

Technical Points





For $\alpha = 0.2$ (distorted BH)

- for each l there are $l + 1$ different values of $\gamma(\omega)$
- $\gamma(\omega)$ gets increased with increasing the deformation α
- for a fixed value of l, $\gamma(\omega)$'s with the max. projection value $m = l$ have the complete transmission at the lowest values of ω.

Discussion of the results

Discussion of the results

We have discussed the scattering problem for d-wave perturbations over the quasi-spherical BH background mostly focusing on the axially symmetric metric potential and on odd (axial) perturbations. We have made the following findings:

- Despite the spherical symmetry breaking, wave equations admit the separation of variables in radial and angular coordinates.
- The effective potential $V_s(r)$ is constructed with replacement of the common angular momentum $l \in \mathbb{Z}$ with some ν, taking generally non-integer values.
- For each value of l there are $l+1$ different eigenvalues ν_{lm} which are depend on the deformation (of the spherical symmetry) parameter α.
- Greybody factors increase with increasing α.
- A preliminary analysis of the QNMs shows that for large values of α the spacetime undergoes instability.