The use of the effective potential for construction of inflationary models with the Gauss-Bonnet term

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MODIFIED GRAVITY MODELS WITH THE GAUSS–BONNET TERM

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The first one is an attempt to connect gravity with quantum physics, at least in a perturbative way, by including quantum correction terms to Einstein's equations.

The second one is an interest to describe the Universe evolution in a more natural way, without the dark energy and the dark matter components, which turn out to be avoidable in the modified models.

The Gauss–Bonnet models are motivated by α' corrections in string theories. The most general Lagrangian density at the next to leading order in the parameter α' reads[1]:

$$L_{string} = -\frac{\lambda}{2}\alpha'\xi(\phi)\left[c_1\mathcal{G} + c_2 G^{\mu\nu}\partial_\mu\phi\partial_\mu\phi + c_3\Box\phi\phi^{;\mu}\phi_{;\mu} + c_4(\phi^{;\mu}\phi_{;\mu})^2\right],$$

where

- $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor;
- $\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss–Bonnet term;
- $\alpha' = \lambda_s^2$, where λ_s is the fundamental string length scale;
- c_i are constants (we will consider the case $c_k = 0$, $k = 2, 3, 4$);
- λ is an additional parameter allowing for different species of string theories, $\lambda = -1/4$ for the Bosonic string and $\lambda = -1/8$ for Heterotic string respectively.

[1] D.J. Gross and J.H. Sloan, Nucl. Phys. B **291** (1987) 41; R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. B **293** (1987) 385.

INFLATIONARY MODELS

The perturbation theory for such types of models has been developed in
C. Cartier, J. c. Hwang and E. J. Copeland, *Evolution of cosmological perturbations in nonsingular string cosmologies*, Phys. Rev. D **64** (2001) 103504 [astro-ph/0106197];
J. c. Hwang and H. Noh, *Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyon: Unified analysis*, Phys. Rev. D **71** (2005) 063536 [gr-qc/0412126]
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Z.K. Guo and D.J. Schwarz, Phys. Rev. D **81**, 123520 (2010) [arXiv:1001.1897]
G. Hikmawan, J. Soda, A. Suroso, and F.P. Zen, Phys. Rev. D **93**, 068301 (2016) [arXiv:1512.00222]
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K. Nozari and N. Rashidi, Phys. Rev. D **95** (2017) 123518 [arXiv:1705.02617]
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MODELS WITH THE GAUSS–BONNET TERM

$$S=\int d^4x\sqrt{-g}\left(U(\phi)R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)-\frac{1}{2}\xi(\phi)\mathcal{G}\right). \quad (1)$$

In the spatially flat FLRW universe with the interval

$$ds^2=\,-\,dt^2+a^2(t)\left(dx_1^2+dx_2^2+dx_3^2\right),$$

one gets the following equations

$$6H^2U+6HU'\dot\phi=\frac{1}{2}\dot\phi^2+V+12H^3\xi'\dot\phi, \quad (2)$$

$$4\left(U-2H\xi'\dot\phi\right)\dot H=\,-\,\dot\phi^2-2\ddot U+2H\dot U+4H^2\left(\ddot\xi-H\dot\xi\right), \quad (3)$$

$$\ddot\phi+3H\dot\phi-6(\dot H+2H^2)U'+V'+12H^2\xi'\left(\dot H+H^2\right)=0, \quad (4)$$

where $H=\dot a/a$ is the Hubble parameter, primes mean the derivatives with respect to ϕ and dots mean the derivatives with respect to the cosmic time.

DE SITTER SOLUTIONS

Let us find de Sitter solutions in the model with the Gauss–Bonnet term. We restrict ourselves to de Sitter solutions with a constant $\phi = \phi_{dS}$. Substituting $\phi = \phi_{dS}$ and $H = H_{dS}$ into Eqs. (2) and (4), we get:

$$6H_{dS}^2 U_{dS} = V_{dS}, \tag{5}$$

$$\xi'_{dS} = \frac{3U_{dS}(2U'_{dS}V_{dS} - V'_{dS}U_{dS})}{V_{dS}^2}. \tag{6}$$

where $V_{dS} = V(\phi_{dS})$, $U_{dS} = U(\phi_{dS})$, and $\xi_{dS} = \xi(\phi_{dS})$.
The value of the Hubble parameter at the de Sitter point is the same as in the corresponding model without the Gauss–Bonnet term:

$$H_{dS}^2 = \frac{V_{dS}}{6U_{dS}}. \tag{7}$$

For arbitrary functions U and V with $VU > 0$, we can choose $\xi(\phi)$ such that the corresponding point becomes a de Sitter solution with the Hubble parameter defined by (7) and the value of $\xi'(\phi_{dS})$ is fixed by (6).

THE EFFECTIVE POTENTIAL

It would be convenient to obtain position and stability of de Sitter solutions using only one combination of three functions: U, V, and ξ.
We restrict ourselves to the case of $U > 0$.
To get this combination (the effective potential) we cast Eqs. (3) and (4) as a dynamical system:

$$\dot{\phi} = \psi,$$

$$\dot{\psi} = \frac{1}{2\left(B - 2H\xi'\psi\right)}\left\{2H\left[3B + 2\xi' V' - 6U'^2 - 6U\right]\psi - 2\frac{V^2}{U}X \right.$$
$$\left. + \left[12\left[\left(U'' + \frac{3}{2}\right)\xi' + U'\xi''\right]H^2 - 24H^4\xi'\xi'' - 3(2U'' + 1)U'\right]\psi^2\right\},$$

$$\dot{H} = \frac{8\left(U' - 2H^2\xi'\right)H\psi - 2\frac{V^2}{U^2}\left(2H^2\xi' - U'\right)X + \left(4H^2\xi'' - 2U'' - 1\right)\psi^2}{4\left(B - 2\xi' H\psi\right)},$$

where

$$B = 3\left(2H^2\xi' - U'\right)^2 + U, \quad X = \frac{U^2}{V^2}\left[12\xi' H^4 - 12U' H^2 + V'\right].$$

We propose such effective potential V_{eff} that

- a de Sitter solution corresponds to
$$V'_{eff}(\sigma_{dS}) = 0$$
- $V''_{eff}(\sigma_{dS}) > 0$ corresponds to stable de Sitter solutions.

We introduce the effective potential $V_{eff}(\phi)$ in the model with the Gauss–Bonnet term, such that

$$V'_{eff}(\phi_{dS}) = X(\phi_{dS}) = 0. \qquad (8)$$

Indeed, let

$$V_{eff} = -\frac{U^2}{V} + \frac{1}{3}\xi. \qquad (9)$$

we get

$$X(\phi_{dS}) = \frac{1}{3}\xi'_{dS} - 2\frac{U'_{dS}U_{dS}}{V_{dS}} + \frac{V'_{dS}U^2_{dS}}{V^2_{dS}} = V'_{eff}(\phi_{dS}) = 0. \qquad (10)$$

So, de Sitter solutions correspond to extremum points of the effective potential V_{eff}.

Stable de Sitter solutions correspond to $V''_{eff} > 0$.

E.O. Pozdeeva, M. Sami, A.V. Toporensky and S.Yu. Vernov, *Phys. Rev. D* **100** (2019) 083527 [arXiv:1905.05085]

MODELS WITH $V = CU^2$

• Let us consider the case $V = CU^2$, where C is a positive constant.
• In this case, a model without the Gauss–Bonnet term transforms to a model with a constant potential in the Einstein frame.
• If the Gauss–Bonnet term is presented, then the function $\xi(\phi)$ plays a role of the effective potential, fully determining the position and stability of the de Sitter solutions, because

$$V_{eff} = -\frac{1}{C} + \frac{1}{3}\xi. \tag{11}$$

• So, values of ϕ_{dS} satisfy the condition $\xi'(\phi_{dS}) = 0$.
A de Sitter solution is unstable at $\xi'' < 0$ and stable at $\xi'' > 0$.
Note that the only difference between minimal and non-minimal coupling cases is that values of the Hubble parameter at de Sitter points $H^2_{dS} = \frac{C}{6}U(\phi_{dS})$, can be different if U is not a constant.

Models with $U = U_0$ and $V = \lambda\phi^4$

Let us consider the following model,

$$U = U_0, \qquad V = \lambda\phi^4, \qquad \xi = \xi_2\phi^{-2} + \xi_4\phi^{-4} + \xi_6\phi^{-6}, \tag{12}$$

with arbitrary constants $U_0 > 0$, λ, ξ_2, ξ_4, and ξ_6.
The condition $V'_{eff}(\phi_{dS}) = 0$ gives the following values of ϕ^2_{dS}:

$$\phi^2_{dS} = \begin{cases} \dfrac{-\beta \pm \sqrt{\beta^2 - 3\xi_2\xi_6}}{\xi_2}, & \text{at} \quad \xi_2 \neq 0, \\ -\dfrac{3\xi_6}{2\beta}, & \text{at} \quad \xi_2 = 0, \end{cases} \tag{13}$$

where $\beta = \xi_4 - 3U_0^2/\lambda$.
Without loss of generality, we consider $\phi_{dS} > 0$ only.
The condition $\phi^2_{dS} > 0$ gives restrictions on values of the parameters.
If $\xi_2 = 0$, then a de Sitter solution exists if and only if $\xi_6/\beta < 0$.

During inflation $H(t)$ is always finite and positive, therefore, it is possible to use the dimensionless parameter $N = \ln(a/a_e)$, where a_e is a constant, as a new measure of time. We fix a_e by the condition $\epsilon_1 = 1$ at $N = 0$. In the case of $U = U_0 = const$, the slow-roll parameters are

$$\begin{aligned}
\epsilon_1 &= -\frac{\dot{H}}{H^2} = -\frac{d\ln(H)}{dN}, \qquad \epsilon_{i+1} = \frac{d\ln|\epsilon_i|}{dN}, \quad i \geqslant 1,\\
\delta_1 &= \frac{2}{U_0}H\dot{\xi} = \frac{2}{U_0}H^2\xi'\frac{d\phi}{dN}, \qquad \delta_{i+1} = \frac{d\ln|\delta_i|}{dN}, \quad i \geqslant 1,
\end{aligned}$$

where we use $d/dt = H\, d/dN$.

The slow-roll conditions $\epsilon_1 \ll 1$, $\epsilon_2 \ll 1$, $\delta_1 \ll 1$, and $\delta_2 \ll 1$ allow to simplify Eqs. (2)–(4):

$$H^2 \simeq \frac{V}{6U_0}, \tag{14}$$

$$\dot{H} \simeq -\frac{\dot{\phi}^2}{4U_0} - \frac{\dot{\xi}H^3}{U_0}, \tag{15}$$

$$\dot{\phi} \simeq -\frac{V' + 12\xi' H^4}{3H}. \tag{16}$$

The use of the effective potential

The effective potential $V_{eff}(\phi) = \frac{1}{3}\xi(\phi) - \frac{U_0^2}{V(\phi)}$ is not defined in the case of $V(\phi) \equiv 0$, but inflationary scenarios are always unstable in this case[2]. Using Eqs. (15) and (16), we get

$$\frac{dH}{dN} \simeq -\frac{H}{U_0} V' V'_{eff}, \qquad (17)$$

$$\frac{d\phi}{dN} \simeq -2\frac{V}{U_0} V'_{eff}. \qquad (18)$$

In terms of the effective potential the slow-roll parameters are as follows:

$$\epsilon_1 = -\frac{1}{2}\frac{d\ln(V)}{dN} = \frac{V'}{U_0}V'_{eff}, \qquad \epsilon_2 = -\frac{2V}{U_0}V'_{eff}\left[\ln(V'V'_{eff})\right]', \qquad (19)$$

$$\delta_1 = -\frac{2V^2}{3U_0^3}\xi' V'_{eff}, \qquad \delta_2 = -\frac{2V}{U_0}V'_{eff}\left[\ln(V^2\xi' V'_{eff})\right]'. \qquad (20)$$

So, $|\epsilon_1| \ll 1$ and $|\delta_1| \ll 1$ if V'_{eff} is small enough.

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[2] G. Hikmawan, J. Soda, A. Suroso, and F.P. Zen, Phys. Rev. D **93**, 068301 (2016) [arXiv:1512.00222].

Inflationary parameters

The tensor-to-scalar ratio r and the spectral index n_s are:

$$r = 8|2\epsilon_1 - \delta_1| = \frac{4}{U_0}\left[\frac{d\phi}{dN}\right]^2 = 16\frac{V^2}{U_0^3}\left(V'_{eff}\right)^2 = -8\frac{V}{U_0^2}\frac{dV_{eff}}{dN}, \quad (21)$$

$$\begin{aligned} n_s =& 1 - 2\epsilon_1 - \frac{2\epsilon_1\epsilon_2 - \delta_1\delta_2}{2\epsilon_1 - \delta_1} = 1 - 2\epsilon_1 - \frac{d\ln(r)}{dN} \\ =& 1 + \frac{d\ln(V/r)}{dN} = 1 + \frac{2}{U_0}\left(2VV''_{eff} + V'V'_{eff}\right). \end{aligned} \quad (22)$$

Moreover, the spectral index n_s can be presented via derivatives of the effective potential only:

$$n_s(N) = 1 - \frac{d}{dN}\left(\ln\left(\frac{dV_{eff}}{dN}\right)\right). \quad (23)$$

The expression for the scalar perturbation amplitude A_s in the leading order approximation is:

$$A_s \approx \frac{H^2}{\pi^2 U_0 r} \approx \frac{V}{6\pi^2 U_0^2 r} = -\frac{1}{48\pi^2 \frac{dV_{eff}}{dN}}. \quad (24)$$

Inflationary models with $\xi(\phi) = C/V(\phi)$

The choice of the function $\xi(\phi) = C/V(\phi)$, where C is a constant, is actively studied.
In this case,

$$V_{eff} = \frac{C - 3U_0^2}{3V}, \tag{25}$$

and the slow-roll parameters are as follows:

$$\epsilon_1 = \frac{\left(3U_0^2 - C\right) V'^2}{3U_0 V^2}, \quad \epsilon_2 = \frac{4\left(C - 3U_0^2\right)\left(VV'' - V'^2\right)}{3U_0 V^2},$$

$$\delta_1 = \frac{2C}{3U_0^2}\epsilon_1\,, \quad \delta_2 = \epsilon_2\,.$$

So, the inflationary parameters are

$$n_s = 1 + \frac{2\left(3U_0^2 - C\right)\left(2VV'' - 3V'^2\right)}{3U_0 V^2}\,. \tag{26}$$

$$r = \frac{16V'^2\left(3U_0^2 - C\right)^2}{9U_0^3 V^2}, \tag{27}$$

In the case $V = V_0\phi^n$, we obtain, taking into account $\epsilon_1(\phi(0)) = 1$,

$$\phi^2(N) = \frac{n(4N-n)(C-3U_0^2)}{3U_0}.$$

$$n_s = 1 + \frac{2(n+2)}{4N-n}, \qquad r = \left|\frac{16n(C-3U_0^2)}{3U_0^2(4N-n)}\right|, \tag{28}$$

In the case of a monomial potential V, adding of the GB term with $\xi = C/V$ does not change n_s, but changes r.
In case $n = 4$ and $\epsilon_1 = \epsilon_2 = \delta_2$, we obtain in the slow-roll approximation

$$r = \left|\frac{16(C-3U_0^2)}{3U_0^2(N-1)}\right|, \qquad n_s = 1 + \frac{3}{N-1}, \tag{29}$$

The Planck observation: $n_s = 0.9649 \pm 0.0042$ at 68% CL, implies that $-96 < N < -75$.

The $\lambda\phi^4$ potential

Let

$$V = \lambda\phi^4, \qquad \xi = \xi_2\phi^{-2} + \xi_4\phi^{-4} + \xi_6\phi^{-6}, \tag{30}$$

with arbitrary constants λ, ξ_2, ξ_4, and ξ_6.

$$V_{eff} = \frac{\xi_2}{3\phi^2} + \frac{\beta}{3\phi^4} + \frac{\xi_6}{3\phi^6}, \tag{31}$$

where $\beta = \xi_4 - 3U_0^2/\lambda$.
The inflationary parameters are as follows:

$$n_s = 1 + \frac{8\lambda(\xi_2\phi^4 + 6\beta\phi^2 + 15\xi_6)}{3U_0\phi^4}, \qquad r = \frac{64\lambda^2\left(\xi_2\phi^4 + 2\beta\phi^2 + 3\xi_6\right)^2}{9U_0^3\phi^6}. \tag{32}$$

In the generic case, when parameters ξ_2, ξ_6, and β are nonzero, analytical solutions cannot be obtained. We use numeric computations at $\lambda = 0.1$, $\xi_6 = -0.1$, $U_0 = 1/2$, and $\beta = -7.4$.
The choice of β is from the fact to keep $A_s \sim 2.1 \times 10^{-9}$.
The parameter ξ_2 is taken in the range $0 \leq \xi_2 \leq 0.5$.

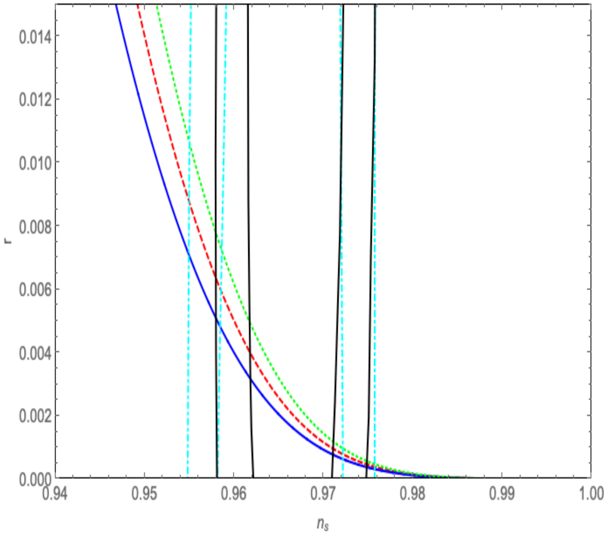

Figure: Blue solid, red dashed and green dot curves correspond to $N = -55$, $N = -60$, and $N = -65$ respectively. The contours correspond to the marginalized joint 68% and 95% CL.

The case of $\beta = 0$

In the case of $\xi_6 \neq 0$ and $\beta = 0$, the inflationary parameters are as follows:

$$n_s = 1 + \frac{8\lambda(\xi_2\phi^4 + 15\xi_6)}{3U_0\phi^4}, \qquad r = \frac{64\lambda^2\left(\xi_2\phi^4 + 3\xi_6\right)^2}{9U_0^3\phi^6}. \tag{33}$$

The solution of Eq. (18) with an additional condition $\epsilon_1(\phi(0)) = 1$ has the following form:

$$\phi(N) = \sqrt[4]{\frac{3\xi_6(3U_0\mathrm{e}^{16\lambda\xi_2 N/(3U_0)} - 8\lambda\xi_2 - 3U_0)}{\xi_2(8\lambda\xi_2 + 3U_0)}} \tag{34}$$

Substituting (34) into expressions (33), we get:

$$n_s = 1 + \frac{8\lambda\xi_2(3U_0\mathrm{e}^{16\lambda\xi_2 N/(3U_0)} + 32\lambda\xi_2 + 12U_0)}{3U_0\left(3U_0\mathrm{e}^{\frac{16\lambda\xi_2 N}{3U_0}} - 8\lambda\xi_2 - 3U_0\right)},$$

$$r = \frac{64\sqrt{3\xi_6}\lambda^2\xi_2^3\mathrm{e}^{\frac{32\lambda\xi_2 N}{3U_0}}\sqrt{(8\lambda\xi_2 + 3U_0)}}{U_0(3U_0\mathrm{e}^{\frac{16\lambda\xi_2 N}{3U_0}} - 8\lambda\xi_2 - 3U_0)(8\lambda\xi_2 + 3U_0)\sqrt{\xi_2^3(3U_0\mathrm{e}^{\frac{16\lambda\xi_2 N}{3U_0}} - 8\lambda\xi_2 - 3U_0)}}$$

The inflationary parameter n_s does not depend on ξ_6, but we cannot put $\xi_6 = 0$, because $\phi(N) \equiv 0$ in this case. In Fig. 2, we show the variation of r with n_s for a particular choice of $\lambda = 0.1$, $\xi_6 = -0.1$, and $U_0 = 1/2$.

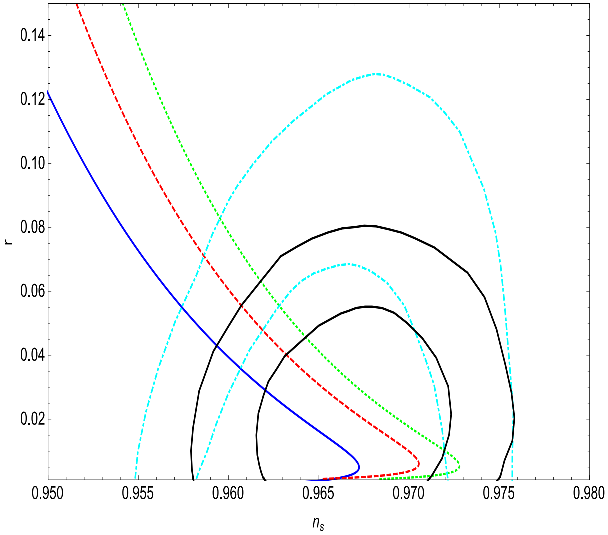

Figure: Parameter space of n_s and r for the model with $V = \lambda\phi^4$ and $\xi = \xi_2\phi^{-2} + \xi_4\phi^{-4} + \xi_6\phi^{-6}$ in the case $\beta = 0$. Blue solid, red dashed, and green dot curves correspond to $N = -55$, $N = -60$, and $N = -65$ respectively. The contours correspond to the marginalized joint 68% and 95% CL.

Conclusions

• We analyze the Einstein–Gauss–Bonnet gravity model:

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{2}\xi(\phi)\mathcal{G}\right),$$

• We have shown that, in the case of $U(\phi) > 0$, it is possible to introduce the effective potential V_{eff} which can be expressed through the coupling function U, the scalar field potential V and the coupling function with the Gauss–Bonnet term ξ:

$$V_{eff} = \frac{1}{3}\xi - \frac{U^2}{V}.$$

• It is convenient to investigate the structure of fixed points using the effective potential, indeed, the stable de Sitter solutions correspond to minima of the effective potential V_{eff}.

• The effective potential V_{eff} plays an important role in the inflationary scenario construction, in particular, in the search of suitable values of model parameters.

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Thank for your attention