From thin to thick domain walls: An example of the $\phi^8$ model
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Abstract
We study an example of higher-order field-theoretic model with an eighth-degree polynomial potential – the $\phi^8$ model. We show that for some certain ratios of constants of the potential the problem of finding kink-type solutions in (1+1)-dimensional space-time reduces to solving algebraic equations. Based on the explicit formulas found for the kink solutions, we show that for certain values of the constants, kink-like solutions with power-law asymptotics arise in the model, describing, in particular, thick domain walls. Objects of this kind could be of interest for modern cosmology.

The $\phi^n$ model
The Lagrangian:
$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi)$$

with the potential in the form of eighth degree polynomial
$$V(\phi) = \frac{1}{2} (\alpha^2 - \beta^2)^2 (\phi^2 - \beta^3)^2,$$

where $\alpha$ and $\beta$ are constants, $0 < \alpha < \beta$.

Algebraic Equations for Kinks

Topological sectors $(a, b)$ and $(-b, -a)$.

An implicit kink solution:
$$x = \frac{1}{2(\beta - \alpha)} \ln \left( \frac{\phi - a}{\phi + a} \right)^{1/n} \left( \frac{b + \phi}{b - \phi} \right)^{1/n}.$$

Denote $b/a = n$ and set $b = 1$, then we get
$$\left( \frac{n \phi - 1}{n \phi + 1} \right)^n \left( \frac{1 + \phi}{1 - \phi} \right) = a_n(x),$$

where
$$a_n(x) = \exp \left[ \frac{2(1 - 1/n)}{x} \right].$$

Topological sector $(-a, a)$:
$$\left( \frac{1 + n \phi}{1 - n \phi} \right)^n \left( \frac{1 - \phi}{1 + \phi} \right) = a_n(x),$$

Case $n = 2$

Topological sectors $(-1, -1/2)$, $(-1/2, 1/2)$ and $(1/2, 1)$:
$$\phi_K(x) = \cos \left( \frac{1}{2} \arccos \left[ \frac{\tanh \left( \frac{3}{4} x \right)}{\frac{3}{4}} \right] \right),$$

Asymptotics of Kinks and Limit $n \to \infty$

Topological sector $(1/3, 1)$ and symmetrical to it $(-1, -1/3)$:
$$\phi^{(1)}_{K}(x) = \begin{cases} \frac{1}{3} \left[ 1 - \frac{1}{3} \frac{\sech \left( \frac{8}{9} x \right)}{\sech \left( \frac{8}{9} x \right)} + \frac{2 \tanh \left( \frac{8}{9} x \right)}{\sqrt{1 - \sech^2 \left( \frac{8}{9} x \right)}} \right], & x < 0, \\ \frac{1}{3} \left[ 1 + \frac{1}{3} \frac{\sech \left( \frac{8}{9} x \right)}{\sech \left( \frac{8}{9} x \right)} + \frac{2 \tanh \left( \frac{8}{9} x \right)}{\sqrt{1 - \sech^2 \left( \frac{8}{9} x \right)}} \right], & x > 0. \end{cases}$$

Topological sector $(-1/3, 1/3)$:
$$\phi^{(2)}_{K}(x) = \begin{cases} \frac{1}{3} \left[ 1 - \frac{1}{3} \frac{\cosh \left( \frac{8}{9} x \right)}{\cosh \left( \frac{8}{9} x \right)} - \frac{2 \coth \left( \frac{8}{9} x \right)}{\sqrt{1 + \cosh^2 \left( \frac{8}{9} x \right)}} \right], & x < 0, \\ \frac{1}{3} \left[ 1 + \frac{1}{3} \frac{\cosh \left( \frac{8}{9} x \right)}{\cosh \left( \frac{8}{9} x \right)} - \frac{2 \coth \left( \frac{8}{9} x \right)}{\sqrt{1 + \cosh^2 \left( \frac{8}{9} x \right)}} \right], & x > 0. \end{cases}$$

In the limit $n \to \infty$ (i.e., for $n \to 0$) at $x \to +\infty$ it can be seen that for any finite $x$ the argument of the exponent tends to zero, which corresponds to the transition from exponential to power-law asymptotic behavior of the kink in the topological sector $(0, 1)$.

Conclusion
We have shown that in the case of a ratio of constants $b/a = n$ equal to positive integers, in order to obtain explicit formulas for kinks, it is necessary to solve an algebraic equation of degree $n + 1$. As an example, we have considered cases of $n = 2$ and 3 and obtained analytical formulas for kinks in all topological sectors of the model. For $n = 3$, the expressions for kinks look rather cumbersome; nevertheless, this is a significant step forward in the study of topological solutions of the $\phi^8$ model.

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