Coordinate-space representation of a charged scalar particle propagator in a constant magnetic field expanded as a sum over the Landau levels

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Introduction

There exist different representations of charged particle propagators in a constant magnetic field. Among the most useful are the Fock-Schwinger proper-time representation, both in the coordinate and momentum spaces, and the momentum-space representation with the expansion over the Landau levels. In this study we derive the missing coordinate-space representation for the propagator of a charged scalar particle as a series over the Landau levels. First, a *modified* Fock-Schwinger method is used to obtain an intermediate expression as a partial Fourier image in *t,y,z*-coordinates expanded over Landau levels. Next, it is symmetrized with respect to *x,y*-coordinates. Finally, the remaining Fourier integrals over *t,z*-coordinates are evaluated. In the resulting expression, each expansion term explicitly decomposes into two factors. The first factor, a sum of Bessel functions, depends only on time and *z*-coordinate, where the *z*-axis is chosen to be a direction of the magnetic field. The second factor, a product of a Laguerre polynomial and a damping exponential, depends on *x,y*-coordinates, which form a plane perpendicular to the direction of magnetic field.

Outline of the modified Fock-Schwinger approach

There are many ways to obtain propagators of charged particles in a constant magnetic field [1], one of which consists in solving the corresponding propagator equation provided by the path integral formalism:

$$H(\partial_X, X)G(X,X') = \delta^4(X-X') \tag{1}$$

In this study we apply our recently published modification [2,3] of the Fock-Schwinger (FS) method [4] to get the solution of (1). First, the propagator $G(X,X')$ is represented as an integral:

$$G(X,X') = -i\int_{-\infty}^{0} d\tau\, U(X,X';\tau) \tag{2}$$

where τ is called the *proper time*. Considering $U(X,X';\tau)$ as some sort of an evolution operator satisfying a Schrödinger-type equation

$$i\partial_\tau U(X,X';\tau) = H(\partial_X,X)U(X,X';\tau) \tag{3}$$

with the appropriate boundary conditions

$$U(X,X';-\infty) = 0 \qquad U(X,X';0) = \delta^4(X-X') \tag{4}$$

one obtains the following expression:

$$U(X,X';\tau) = exp(-i\tau[H(\partial_X,X)+i\varepsilon])\,\delta^4(X-X') \tag{5}$$

The modified Fock-Schwinger (MFS) method consists in the direct evaluation of the exponential operator action on the delta-function. In order to do so, an appropriate representation of the delta-function as an integral and/or series should be chosen:

$$\delta^4(X-X') = \sum_\lambda \int \psi_\lambda(X)\psi_\lambda(X') \tag{6}$$

where $\psi_\lambda(X)$ is the eigenvector of the **H** operator:

$$H\psi_\lambda(X) = \lambda\,\psi_\lambda(X) \tag{7}$$

This leads to the following simplification:

$$G(X,X') = -i\int_{-\infty}^{0} d\tau \sum_\lambda \int e^{-i\tau[H(\lambda)+i\varepsilon]}\psi_\lambda(X)\psi_\lambda(X') \tag{8}$$

Next, the exponential part is integrated out:

$$G(X,X') = \sum_\lambda \int \frac{\psi_\lambda(X)\psi_\lambda(X')}{H(\lambda)+i\varepsilon} \tag{9}$$

The resulting expression is an integral/series over the continuous/discrete quantum numbers, and does not contain the proper time parameter.

The MFS method is one of the fastest ways to obtain propagators, especially in the external fields for particles with spin. We will apply it in the next section to get an intermediate form of the propagator which will serve as a starting point for the calculation of the coordinate-space representation expanded in a sum over Landau levels.

Calculation of a charged scalar particle propagator

We apply the MFS method to solve the propagator equation for a charged scalar particle:

$$\left[\left(i\partial_\mu - eQA_\mu\right)^2 - m^2\right]G(X,X') = \delta^4(X-X') \tag{10}$$

For convenience, we choose the Landau gauge: $A^\mu = (0,0,Bx,0)$. With this choice of gauge, the **H** operator obtains the form:

$$H(\partial_X,X) = \left(i\partial_\mu - eQA_\mu\right)^2 - m^2 = p_0^2 - p_z^2 - m^2 + \beta\left(d_\xi^2 - \xi^2\right) \tag{11}$$

where the notations are used: $\beta = eB$, $\xi = \sqrt{\beta}\left(x - Q\frac{p_y}{\beta}\right)$.

The substitutions $i\partial_0 \to p_0$, $-i\partial_y \to p_y$, $-i\partial_z \to p_z$ are justified by the appropriate representation of delta-function

$$\delta^4(X-X') = \sqrt{\beta}\sum_{n=0}^{\infty}\int \frac{d^3p_{\parallel,y}}{(2\pi)^3}\, e^{-i\left(p(X-X')\right)_{\parallel,y}} V_n(\xi)V_n(\xi') \tag{12}$$

where $\parallel$ stands for *t,z*-coordinates, V_n are the quantum harmonic oscillator (QHO) eigenfunctions

$$V_n(\xi) = \left(2^n\, n!\,\sqrt{\pi}\,\right)^{-1/2} exp\left(-\xi^2/2\right) H_n(\xi)$$

and H_n are the Hermite polynomials. Using the formula (8) and the QHO eigenvalue equation

$$\left(d_\xi^2 - \xi^2\right) V_n(\xi) = -(2n+1)V_n(\xi)$$

one obtains the following representation of the propagator:

$$G(X,X') = (-i)\sqrt{\beta}\sum_{n=0}^{\infty}\int \frac{d^3p_{\parallel,y}}{(2\pi)^3}\int_{-\infty}^{0} d\tau\; e^{-i\tau[p_\parallel^2 - M_n^2 + i\varepsilon]}\; e^{-i\left(p(X-X')\right)_{\parallel,y}} V_n(\xi)V_n(\xi') \tag{13}$$

where $M_n^2 = m^2 + \beta(2n+1)$. A straightforward evaluation of the integral over τ leads to:

$$G(X,X') = \sqrt{\beta}\sum_{n=0}^{\infty}\int \frac{d^3p_{\parallel,y}}{(2\pi)^3}\; \frac{e^{-i\left(p(X-X')\right)_{\parallel,y}}}{p_\parallel^2 - M_n^2 + i\varepsilon} V_n(\xi)V_n(\xi') \tag{14}$$

This form of the propagator, however, is not invariant with respect to rotations in the *x,y*-plane and, thus, does not exhibit the internal symmetry of the problem. In order to symmetrize it, we use the following formula [5]:

$$I_{n,n'} = \int_{-\infty}^{\infty} du\; e^{-u^2} H_n(u+a)H_{n'}(u+b) = 2^{n'}\sqrt{\pi}\, n!\, b^{n'-n}\, L_n^{(n'-n)}(-2ab) \tag{15}$$

where $n' \geq n$ and $L_n^{(m)}$ are associated Laguerre polynomials.

Making an appropriate change of variables and integrating over p_y leads to the following symmetrized representation:

$$G(X,X') = \frac{\beta}{2\pi}e^{i\Phi}\sum_{n=0}^{\infty} e^{-\frac{\beta X_\perp^2}{4}} L_n\left(\frac{\beta X_\perp^2}{2}\right)\int \frac{d^2p_\parallel}{(2\pi)^2}\frac{e^{-i\left(p(X-X')\right)_{\parallel,}}}{p_\parallel^2 - M_n^2 + i\varepsilon} \tag{16}$$

Here, $\perp$ subscript stands for the *x,y*-coordinates. We also note an overall non-invariant phase factor $e^{i\Phi}$ where $\Phi = -\frac{Q\beta}{2}(x-x')(y-y')$. Aside from this phase, the summation terms decompose into 2 factors. The first one depends only on the coordinates in the *x,y*-plane which is perpendicular to the direction of magnetic field. The second factor (the Fourier integral)

$$I_\parallel = \int \frac{d^2p_\parallel}{(2\pi)^3}\frac{e^{-i\left(p(X-X')\right)_{\parallel,}}}{p_\parallel^2 - M_n^2 + i\varepsilon} \tag{17}$$

depends only on the *t,z*-coordinates and allows for further simplification. The integration proceeds much like in the case of a free field [6] and relies on the following integral identity [5]:

$$\int_0^\infty dx\, cos(bx)\frac{e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} = K_0\left(\gamma\sqrt{\beta^2+b^2}\right), \qquad Re\,\beta > 0, Re\,\gamma > 0, b > 0$$

where K_0 is the modified Bessel function of the second kind.

The resulting expression can be written as a distribution (generalized function):

$$I_\parallel = \lim_{\epsilon\to 0^+}\left[\frac{-i}{2\pi}K_0\left(M_n\sqrt{-x_\parallel^2 + i\epsilon}\right)\right] \tag{18}$$

We keep this expression for $x_\parallel^2 < 0$. For the case when $x_\parallel^2 > 0$, we transform it by applying another useful identity from [5]:

$$K_\nu(z) = -\frac{i\pi}{2}e^{-i\pi\nu/2}H_{-\nu}^{(2)}\left(ze^{-i\pi/2}\right) \tag{19}$$

where $H_\nu^{(2)}$ are the Hankel functions of the second kind.

The final expression for the coordinate-space representation of the propagator reads:

$$G(x,x') = \frac{\beta}{4\pi^2}e^{i\Phi}\lim_{\epsilon\to 0^+}\sum_{n=0}^{\infty} e^{-\frac{\beta x_\perp^2}{4}} L_n\left(\frac{\beta x_\perp^2}{2}\right)\left\{K_0\left(M_n\sqrt{-x_\parallel^2 + i\epsilon}\right)\theta\left(-x_\parallel^2\right) - \frac{i\pi}{2}H_0^{(2)}\left(M_n\sqrt{x_\parallel^2 - i\epsilon}\right)\theta\left(x_\parallel^2\right)\right\} \tag{20}$$

Conclusion

Using the modified Fock-Schwinger method we were able to rapidly obtain the symmetrized representation of the propagator, which allowed for a straightforward evaluation of the remaining Fourier integral. The final expression in the coordinate space as a series over Landau levels decomposed into several important parts. Each series term corresponds to a particular Landau level and consists of two factors. The first factor depends only on the coordinates in the x,y-plane which is perpendicular to the direction of magnetic field, and is invariant with respect to rotations in this plane. The damping exponential ensures that the propagation in the x,y-plane is localized. The second factor describes propagation in the t,z-plane and is similar to the case of free field, however, in Minkowski 2-space instead of 4-space. We also note the non-invariant overall phase factor $e^{i\Phi}$ which naturally arises when considering propagators of charged particles in a constant magnetic field.

Acknowledgments

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