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Energy interval (1S-2S) in muonic ions of lithium, beryllium and boron

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Contents

The task of the experiments of the CREMA Collaboration (Charge Radius Experiment with Muonic Atoms) beginning with 2010 is the measurement the fine and hyperfine structure of the spectrum in light muonic atoms (muonic hydrogen, muonic deuterium, muonic helium ions ...); determination of the charge radii of the proton, deuteron, helion, alpha particle with an accuracy of 0.0005 fm.

1. The measurement of transition frequency $(2S^{F=1}_{1/2}-2P^{F=2}_{3/2})$ in 2010 [1]
2. The measurement of two transition frequencies $(2S^{F=1}_{1/2}-2P^{F=2}_{3/2})$ and $(2S^{F=0}_{1/2}-2P^{F=1}_{3/2})$ in muonic hydrogen, the measurement of hyperfine structure of 2S state in 2013 [2]
3. The measurement of three transition frequencies between levels 2P and 2S in muonic deuterium: $(2S^{F=3/2}_{1/2}-2P^{F=5/2}_{3/2})$, $(2S^{F=1/2}_{1/2}-2P^{F=3/2}_{3/2})$, $(2S^{F=1/2}_{1/2}-2P^{F=1/2}_{3/2})$ [3]

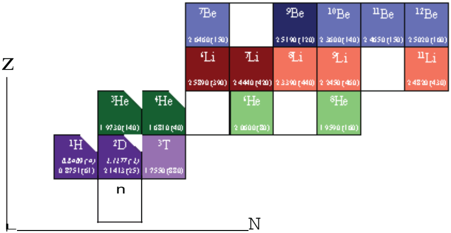
[1] R. Pohl et al., Nature **466**, 213 (2010).

[2] A. Antognini et al., Science **339**, 417 (2013).

[3] R. Pohl et al., Science **353**, 669 (2016).

One of the future tasks of CREMA is to extend the laser spectroscopy experiments with muonic systems to the elements of lithium and beryllium... and to improve the values of charge radii and the Zemach radii of light nuclei

- S. Schmidt et al., The next generation of laser spectroscopy experiments using light muonic atoms arXiv:1808.07240.
- R. Pohl, Laser Spectroscopy of Muonic Atoms and Ions, JPS Conf. Proc. 18, 011021 (2017).



In last years we calculated different energy intervals in the spectrum of muonic lithium, beryllium and boron which can be studied in experiment:

Lamb shift in muonic ions of lithium, beryllium, and boron [1]

Hyperfine structure of S states in muonic ions of lithium, beryllium, and boron [2]

Hyperfine structure of P states in muonic ions of lithium, beryllium, and boron [3]

[1] A. A. Krutov et al., Phys. Rev. A **94**, 062505 (2016).

[2] A. E. Dorokhov et al., Phys. Rev. A **98**, 042501 (2018).

[3] A. E. Dorokhov et al., Phys. Rev. A **100**, 062513 (2019).

The aim of present work is to extend our approach to the calculation of energy interval (2S-1S) for $(\mu^7_3 Li)$, $(\mu^9_4 Be)$, $(\mu^{11}_5 B)$.

The interval (2S-1S) for the electronic hydrogen and deuterium atom was measured to a high degree of accuracy:

$$\nu_{2S-1S}(H) = 2466061413187035(10) Hz, \quad \delta = 4.06 \cdot 10^{-15} \tag{1}$$

Our approach to the calculation of (2S-1S) energy interval is based on the quasipotential method in quantum electrodynamics:

1. The Schrödinger equation for the bound state wave function
2. The Breit Hamiltonian for basic contribution to energy spectrum.

The structure of S-levels is determined by the expression:

$$E_n = m_1 + m_2 - \frac{\mu(Z\alpha)^2}{2n^2} - \frac{\mu(Z\alpha)^4}{2n^3}\left[1 - \frac{3}{4n} + \frac{\mu^2}{4m_1 m_2 n}\right] - \frac{m_1(Z\alpha)^6}{16n^6}(2n^3 + 6n^2 - 12n + 5). \quad (2)$$

The Coulomb wave functions of $1S$- and $2S$-states have the form:

$$\psi_{100}(r) = \frac{W^{3/2}}{\sqrt{\pi}} e^{-Wr}, \quad W = \mu Z\alpha, \quad (3)$$

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-Wr/2}\left(1 - \frac{Wr}{2}\right). \quad (4)$$

Next, we investigate a number of basic corrections to the structure of S-states in order to obtain reliable total result. Numerical values of different corrections are presented for definiteness with the accuracy 10^{-2} meV.

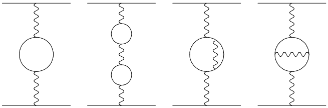


Figure: Effects of one-loop and two-loop VP in 1γ interaction

One-loop VP correction to the Coulomb potential is

$$V^C_{vp}(r) = \frac{\alpha}{3\pi}\int_1^\infty d\xi\rho(\xi)\left(-\frac{Z\alpha}{r}e^{-2m_e\xi r}\right), \quad \rho(\xi) = \frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}. \quad (5)$$

In first order perturbation theory we obtain the shifts of $1S$ and $2S$ states ($b_1 = m_e/W$, $W = \mu Z\alpha$, Z is the nucleus charge):

$$\Delta E_{vp}(1S) = -\frac{4\mu(Z\alpha)^2\alpha}{3\pi}\frac{\sqrt{b_1{}^2-1}\left(12\pi b_1{}^3 - 24b_1{}^2 + 9\pi b_1 - 22\right) - 6\left(4b_1{}^4 + b_1{}^2 - 2\right)\sec^{-1}(b_1)}{6\sqrt{b_1{}^2-1}}, \quad (6)$$

$$\Delta E_{vp}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi}\Big[\frac{(b_1(b_1(16b_1(b_1(3b_1(56b_1(\pi b_1-1)-25\pi)+68)+6\pi)-49)+9\pi)-7)}{3\left(4b_1{}^2-1\right)^2} - \quad (7)$$

$$-\frac{i\left(3584b_1{}^8 - 2048b_1{}^6 + 300b_1{}^4 + 10b_1{}^2 - 1\right)\ln\left(-\frac{2ib_1}{\sqrt{4b_1{}^2-1}-i}\right)}{\left(4b_1{}^2-1\right)^{5/2}}\Big].$$

In the case of fourth-order polarization operator we can construct the interaction potential using the substitution in the photon propagator:

G. Källen and A. Sabry, Mat. Fys. Medd. Dan. Vid. Selesk. **29**, No. 17, 1 (1955), in *Portrait of Gunnar Källen*, edited by C. Jarlskog, Springer, Switzerland, p. 555 (2014).

$$\frac{1}{k^2}\to\frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{4m_e^2+k^2(1-v^2)}, \tag{8}$$

$$f(v)=v\Big\{(3-v^2)(1+v^2)\left[Li_2\left(-\frac{1-v}{1+v}\right)+2Li_2\left(\frac{1-v}{1+v}\right)+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}-\ln\frac{1+v}{1-v}\ln v\right]$$
$$+\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]+\frac{3}{8}v(5-3v^2)\Big\},$$

$Li_2(z)$ is the Euler dilogarithm. In coordinate representation the potential is

$$\Delta V^C_{1\gamma,2-loop\ vp}(r)=-\frac{2Z\alpha}{3r}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{1-v^2}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}. \tag{9}$$

Another loop-after-loop potential in momentum representation is equal to

$$V^C_{vp-vp}(k^2) = -4\pi(Z\alpha)\frac{\alpha^2}{9\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\frac{k^2}{(k^2+4m_e^2\xi^2)(k^2+4m_e^2\eta^2)} = \quad (10)$$

$$= -\frac{2\alpha^2(Z\alpha)}{9\pi}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\Big\{\frac{1}{k^2+4m_e^2\xi^2}+\frac{1}{k^2+4m_e^2\eta^2}-\frac{(\xi^2+\eta^2)}{(\eta^2-\xi^2)}$$

$$\Big[\frac{1}{k^2+4m_e^2\xi^2}-\frac{1}{k^2+4m_e^2\eta^2}\Big]\Big\}.$$

After the Fourier transform we obtain the following sum of the Yukawa potentials distributed with some density:

$$V^C_{1\gamma,vp-vp}(r) = \frac{\alpha^2}{9\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\left(-\frac{Z\alpha}{r}\right)\frac{1}{(\xi^2-\eta^2)}(\xi^2 e^{-2m_e\xi r}-\eta^2 e^{-2m_e\eta r}). \quad (11)$$

The contribution of six-order polarization operator can be divided into two parts with one fermion cycle and two fermion cycles. The following expression obtained by Baikov and Kinoshita is useful (1 fermion cycle)

$$\Pi_3^{(1)}(z) = \tilde{\Pi}_3^{(1)}(z) - 4\Pi_2(z) - (1-z)G(z)\left(\frac{9}{4}G(z) + \frac{31}{16} + \frac{229}{32z} + \frac{229}{32z} + \frac{173}{96}\right), \qquad (12)$$

$$\tilde{\Pi}_3^{(1)}(z) = \tilde{\Pi}_3^{(1)}(z)(-\infty) + \frac{(1+\omega)^2}{(1-\omega)}\frac{(\tilde{a}_0 + \tilde{a}_1\omega + \tilde{a}_2\omega + \tilde{a}_3\omega)}{(\tilde{b}_0 + \tilde{b}_1\omega + \tilde{b}_2\omega + \tilde{b}_3\omega)},$$

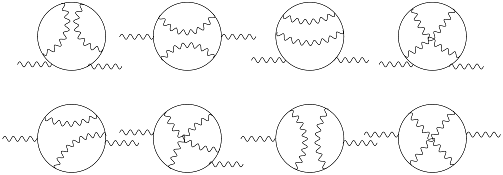


Figure: Three loop VP effects with one fermion cycle in 1γ interaction

The coefficients of Pade approximation are written in

T. Kinoshita and M. Nio, Phys. Rev. D **60**, 053008 (1999).

P. A. Baikov and D. J. Broadhurst, in *New Computing Techniques in Physics Research IV*, edited by B. Denby and D. Perrei-Gallix, World Scientifi Puplishing Co., Singapore, p. 167 (1995).

General formula for the energy interval (2S-1S) takes the form:

$$\Delta E_{vp-8}(2S-1S) = \frac{2}{\pi}\mu(Z\alpha)^2\left(\frac{\alpha}{\pi}\right)^3\int_0^\infty \frac{s^2(104+87s^2+51s^4+14s^6)}{(1+s^2)^4(4+s^2)^2}\Pi_3^{(1)}\left(\frac{s^2}{4b_1^2}\right), \qquad (13)$$

and numerical values are equal

$$\Delta E_{vp-8}(2S-1S) = \begin{cases} 0.1282\ meV, & (\mu Li), \\ 0.1495\ meV, & (\mu Be), \\ 0.1214\ meV, & (\mu B). \end{cases} \qquad (14)$$

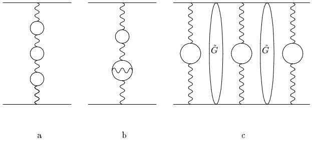


Figure: Effects of three-loop vacuum polarization in one-photon interaction and third-order perturbation theory

The remaining three-loop contributions with successive loops can be calculated in the same way as the two-loop ones, by constructing the interaction potentials. General expressions for these potentials in the coordinate representation are:

$$V^C_{vp-vp-vp}(r) = -\frac{Z\alpha}{r}\frac{\alpha^3}{(3\pi)^3}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\int_1^\infty \rho(\zeta)d\zeta\times \tag{15}$$

$$\times\left[e^{-2m_e\zeta r}\frac{\zeta^4}{(\xi^2-\zeta^2)(\eta^2-\zeta^2)}+e^{-2m_e\xi r}\frac{\xi^4}{(\zeta^2-\xi^2)(\eta^2-\xi^2)}+e^{-2m_e\eta r}\frac{\eta^4}{(\xi^2-\eta^2)(\zeta^2-\eta^2)}\right],$$

$$V^C_{vp-2-loop\ vP} = -\frac{4\mu\alpha^3(Z\alpha)}{9\pi^3 r}\int_1^\infty \rho(\xi)d\xi\int_1^\infty\frac{f(\eta)d\eta}{\eta}\left[e^{-2m_e\eta r}\frac{\eta^2}{\eta^2-\xi^2}-e^{-2m_e\xi r}\frac{\xi^2}{\eta^2-\xi^2}\right]. \tag{16}$$

Corrections in the energy spectrum corresponding to these interactions were presented in integral form over three spectral parameters and calculated numerically.

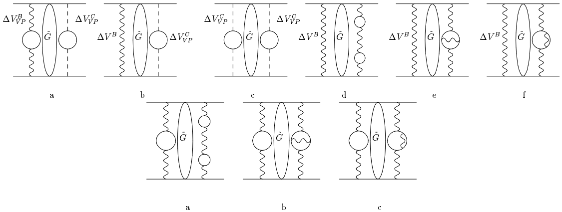


Figure: One-loop, two-loop and three-loop VP effects in second order PT. $\tilde{G}$ is the reduced Coulomb Green function.

Effects of vacuum polarization change also the Breit Hamiltonian:

$$\Delta V^B_{vp}(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi \sum_{i=1}^{3}\Delta V^B_{i,vp}(r), \tag{17}$$

$$\Delta V^B_{1,vp} = \frac{Z\alpha}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\left[4\pi\delta(\mathbf{r}) - \frac{4m_e^2\xi^2}{r}e^{-2m_e\xi r}\right], \tag{18}$$

$$\Delta V^B_{2,vp} = -\frac{Z\alpha m_e^2\xi^2}{m_1 m_2 r}e^{-2m_e\xi r}(1-m_e\xi r), \tag{19}$$

$$\Delta V^B_{3,vp} = -\frac{Z\alpha}{2m_1 m_2}p_i\frac{e^{-2m_e\xi r}}{r}\left[\delta_{ij}+\frac{r_i r_j}{r^2}(1+2m_e\xi r)\right]p_j. \tag{20}$$

Corresponding terms will contribute to the energy spectrum in order $\alpha(Z\alpha)^4$ and corrections are large.

A decrease in the value of the Bohr radius of the orbits in muonic atoms in comparison with electronic ones leads to the fact that the wave function of the muon overlaps strongly with the region of the nucleus. Expanding the charge form factor of the nucleus at small momentum transfers, we find that in the leading order the effect of the structure of the nucleus is determined in the energy spectrum by the following correction proportional to the square of the charge radius r_N^2 (the *str* subscript denotes here and below a correction for the nuclear structure):

$$\Delta E_{str}(2S-1S) = -\frac{7\mu^3(Z\alpha)^4}{12} < r_N^2 >= \begin{cases} -3868.1341\ r_{Li}^2\ meV, & (\mu Li), \\ -12355.5147\ r_{Be}^2\ meV, & (\mu Be), \\ -30369.8008\ r_B^2\ meV, & (\mu B), \end{cases} \quad (21)$$

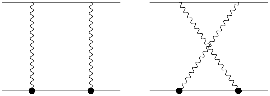


Figure: Nuclear structure corrections of order $(Z\alpha)^5$. Bold point on the diagram denotes the nucleus vertex operator.

The next most important correction is the nuclear structure correction of order $(Z\alpha)^5$ of two-photon exchange amplitudes, which is expressed in terms of the electromagnetic form factors of the nucleus. The shift of the S levels is determined in integral form:

$$\Delta E^{2\gamma}_{str}(nS) = -\frac{\mu^3(Z\alpha)^5}{\pi n^3}\delta_{l0}\int_0^\infty \frac{dk}{k}V_{2\gamma}(k),\;\; V_{2\gamma}(k) = \frac{2(F_1^2-1)}{m_1 m_2} + \frac{8m_1[F_2(0)+4m_2^2F_1'(0)]}{m_2(m_1+m_2)k} + \tag{22}$$

$$+\frac{k^2}{2m_1^3m_2^3}\left[2(F_1^2-1)(m_1^2+m_2^2)+4F_1F_2m_1^2+3F_2^2m_1^2\right]+\frac{\sqrt{k^2+4m_1^2}}{2m_1^3m_2(m_1^2-m_2^2)k}\times$$

$$\left\{k^2[2(F_1^2-1)m_2^2+4F_1F_2m_1^2+3F_2^2m_1^2]-8m_1^4F_1F_2+\frac{16m_1^4m_2^2(F_1^2-1)}{k^2}\right\}-$$

$$-\frac{\sqrt{k^2+4m_2^2}\,m_1}{2m_2^3(m_1^2-m_2^2)k}\left\{k^2[2(F_1^2-1)+4F_1F_2+3F_2^2]-8m_2^2F_1F_2+\frac{16m_2^4(F_1^2-1)}{k^2}\right\},$$

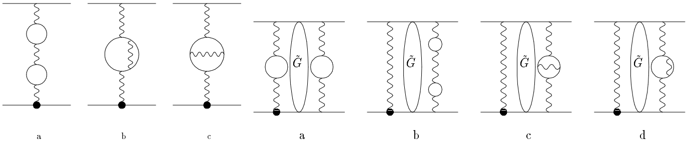


Figure: Nuclear structure and two-loop VP effects in one-photon interaction and second order PT. The bold point in the diagram denotes the vertex operator of the nucleus.

The corrections for the two-loop vacuum polarization taking into account the structure of the nucleus were calculated also. The particle interaction operators corresponding to these amplitudes are constructed in integral form:

$$\Delta V^{vp-vp}_{str}(r) = \frac{2}{3} Z\alpha < r_N^2 > \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times \tag{23}$$

$$\times \left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(\xi^2-\eta^2)}\left(\xi^4 e^{-2m_e\xi r} - \eta^4 e^{-2m_e\eta r}\right)\right],$$

$$\Delta V^{2-loop\ vp}_{str}(r) = \frac{4}{9} Z\alpha < r_N^2 > \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{f(v)dv}{1-v^2}\left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right]. \tag{24}$$

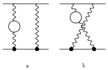


Figure: Nuclear structure and vacuum polarization effects in two-photon exchange diagrams.

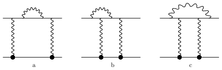


Figure: Two-photon exchange amplitudes with radiative corrections to the muon line, contributing to the order of $\alpha(Z\alpha)^5$.

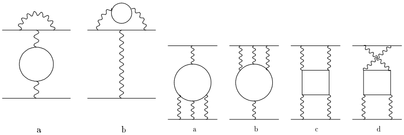


Figure: Radiative correction with vacuum polarization effect and light-by-light scattering amplitudes

Table: Basic parameters of the nucleus lithium, beryllium and boron.

Nucleus	Spin	Mass, GeV	Magnetic dipole moment, nm	Charge radius, fm	Quadrupole moment, fm^2	Magnetic octupole moment , $nm \cdot fm^2$
^7_3Li	3/2	6.53383	3.256427(2)	2.4440 ± 0.0420	-4.06(8)	7.5
^9_4Be	3/2	8.39479	-1.177432(3)	2.5190 ± 0.0120	5.29(4)	4.1
$^{11}_5B$	3/2	10.25510	0.8220473(6)	2.4060 ± 0.0294	4.07(3)	7.8

If we do not fix the numerical values of the corrections to the nucleus structure, then the complete results from the Table can be presented as:

$$\Delta E^{tot}(2S-1S) = \begin{cases} 18744.95385 - 3.868134141\ r^2_{Li}\ eV, & (\mu^7_3 Li), \\ 33471.83067 - 12.355514666\ r^2_{Be}\ eV, & (\mu^9_4 Be), \\ 52459.12345 - 30.369800775\ r^2_{B}\ eV, & (\mu^{11}_5 B). \end{cases} \quad (25)$$

Thus, a precision measurement of the $(2S-1S)$ transition frequency can give new, more accurate values of the charge radii of light nuclei. So, for example, measuring the $(2S-1S)$ shift in muonic ions with a relative error of 1-2 ppm will reduce the error in determining the nucleus charge radius to 0.001 fm.

No.	Contribution to $(2S-1S)$ interval	$(\mu\ {}^7_3Li)$, eV	$(\mu\ {}^9_4Be)$, eV	$(\mu\ {}^{11}_5B)$, eV
1	Fine structure correction	18689.18242	33345.71884	52226.22104
2	VP correction in 1γ interaction of order $\alpha(Z\alpha)^2$	54.48514	120.68223	219.60409
3	Muon vacuum polarization contribution in 1γ interaction of order $\alpha(Z\alpha)^4$	0.01221	0.03867	0.09422
4	Wichman-Kroll correction	-0.00191	-0.00672	-0.01748
5	Light-by-light correction	0.00060	0.00171	0.00375
6	Two-loop VP contribution in 1γ interaction of order $\alpha^2(Z\alpha)^2$	0.39434	0.87206	1.59189
7	Three-loop VP contribution in 1γ interaction of order $\alpha^3(Z\alpha)^2$	0.00140	0.00342	0.00673
8	Relativistic corrections with account one-loop VP in FOPT	-0.03750	-0.13425	-0.35790
9	Relativistic corrections with account of two-loop VP in FOPT	-0.00013	-0.00047	-0.00120
10	Relativistic corrections with account one-loop VP in SOPT	0.06190	0.22387	0.60036
11	Relativistic corrections with account two-loop VP in SOPT	0.00008	0.00019	0.00027
12	Two-loop VP correction in second order PT of order $\alpha^2(Z\alpha)^2$	0.11962	0.29607	0.58110
13	Three-loop VP correction in second order PT of order $\alpha^3(Z\alpha)^2$	0.00173	0.00432	0.00860
14	Three-loop VP correction in third order PT of order $\alpha^3(Z\alpha)^2$	0.00048	0.00143	0.00318
15	Nuclear structure correction of order $(Z\alpha)^4$	-23.10 $\pm$0.80	-78.40 $\pm$0.75	-175.81 $\pm$4.32

1	2	3	4	5
16	Nuclear structure correction with account of one-loop VP of order $\alpha(Z\alpha)^4$	-0.28895	-1.11763	-2.74800
17	Nuclear structure correction with account of two-loop VP of order $\alpha^2(Z\alpha)^4$	-0.00324	-0.01304	-0.03341
18	Nuclear structure correction from 2γ amplitudes of order $(Z\alpha)^5$	1.24	5.78	15.52
19	Nuclear structure and VP correction from 2γ interaction of order $\alpha(Z\alpha)^5$	0.019	0.089	0.236
20	Radiative corrections in muon line with nuclear structure of order $\alpha(Z\alpha)^5$	0.011	0.050	0.139
21	Nuclear polarizability correction	0.147 $\pm$0.028	0.574 $\pm$0.112	0.721 $\pm$0.147
22	Recoil correction of order $(Z\alpha)^5$	-0.01045	-0.03322	-0.08080
23	Recoil correction of order $(Z\alpha)^6$	0.00012	0.00052	0.00163
24	Muon self-energy correction and muon form factors	-0.35367	-1.02434	-2.32300
25	Radiative corrections with recoil of order $\alpha(Z\alpha)^5$ and corrections of nucleus form factors of order $Z^2\alpha(Z\alpha)^4$	-0.00081	-0.00248	-0.00588
26	Nuclear structure corrections of order $(Z\alpha)^6$	-0.03153	-0.18944	-0.68097
27	Muon form factors contribution $F_1'(0)$, $F_2(0)$	-0.00109	-0.00347	-0.00853
28	VP correction with muon self energy	-0.00193	-0.00621	-0.01518
29	Hadronic VP contribution	0.00802	0.02561	0.06294
30	Summary contribution	18721.85385	33393.43067	52283.31345

Thank you for attention!