Quantum Simulation Of Entangled Oscillating Neutrinos (ID-892)

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Overview

Introduction

- The neutrino flavor states $|\nu_\alpha\rangle$ $(\alpha = e, \mu, \tau)$ are linear superposition of mass eigenstates $|\nu_j\rangle$ $(j = 1, 2, 3)$: $|\nu_\alpha\rangle = \Sigma_j U_{\alpha j} |\nu_j\rangle$, where $U_{\alpha j}$ are the elements of the lepton mixing matrix known as PMNS (Pontecorvo-Maki- Nakagawa-Sakita) matrix such that

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix} \tag{1}$$

- The time evolution follows $|\nu_\alpha(t)\rangle = \Sigma_j e^{-iE_j t} U_{\alpha j} |\nu_j\rangle$, where E_j is the energy associated with the mass eigenstates $|\nu_j\rangle$. This is a superposition state. Therefore, we expect quantum entanglement [1].

[1] M.Blasone et al.Phys.Rev.D 77 (2007) no.9, 096002

- In the three mode basis, we identify each flavor state $(\alpha = e, \mu, \tau)$ at t=0 as :

$$
\begin{aligned}
|\nu_e\rangle &= |1\rangle_e \otimes |0\rangle_\mu \otimes |0\rangle_\tau \equiv |100\rangle_e \, , \\
|\nu_\mu\rangle &= |0\rangle_e \otimes |1\rangle_\mu \otimes |0\rangle_\tau \equiv |010\rangle_\mu \, , \\
|\nu_\tau\rangle &= |0\rangle_e \otimes |0\rangle_\mu \otimes |1\rangle_\tau \equiv |001\rangle_\tau \, .
\end{aligned}
$$

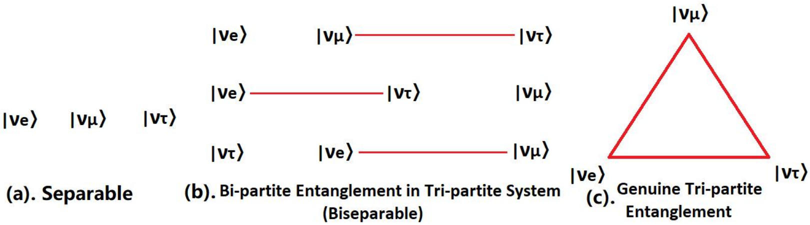

Figure: Different possible ways of visualising three mode (flavor) entanglement

Bi-partite entanglement in two-flavor neutrino oscillations

- In two-flavor ($\nu_\alpha \to \nu_\beta$) mixing, $|\nu_e(t)\rangle = \tilde{U}_{ee}(t)\,|10\rangle_e + \tilde{U}_{e\mu}(t)\,|01\rangle_\mu$, where $|10\rangle$ and $|01\rangle$ are two-qubit states [2],

$$\begin{aligned} |\nu_e\rangle &= |1\rangle_e \otimes |0\rangle_\mu \equiv |10\rangle_e\,, \\ |\nu_\mu\rangle &= |0\rangle_e \otimes |1\rangle_\mu \equiv |01\rangle_\mu\,, \end{aligned}$$

- The appearance (P_a) and disappearance (P_d) probabilities are:

$$P_a = |\tilde{U}_{e\mu}(t)|^2 = \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos\left(\frac{\Delta m^2 t}{2E}\right)$$

$$\text{and} \quad P_d = |\tilde{U}_{ee}(t)|^2 = 4\sin^2\theta\cos^2\theta\sin^2\left(\frac{\Delta m^2 t}{4E}\right). \qquad (2)$$

[2] M.Blasone et al.EPL **85** (2009), 50002

- The density matrix for $|\nu_e(t)\rangle$ can be expressed as,

$$\rho^{e\mu}(t) = |\nu_e(t)\rangle\langle\nu_e(t)| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\tilde{U}_{ee}(t)|^2 & \tilde{U}_{ee}(t)\tilde{U}^*_{e\mu}(t) & 0 \\ 0 & \tilde{U}_{e\mu}(t)\tilde{U}^*_{ee}(t) & |\tilde{U}_{e\mu}(t)|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

- Positive Partial Transpose (PPT) criterion is a condition for determining entanglement in bi-partite system. It states that if the partial transposition $\rho^{T_e}_{pq,rs}(t) = \rho^{e\mu}_{rq,ps}(t)$ or $\rho^{T_\mu}_{pq,rs}(t) = \rho^{e\mu}_{ps,rq}(t)$ of a density matrix is a positive operator with all positive eigenvalues then the system is unentangled. If the system has even one negative eigenvalues then it is entangled [3] .

[3] P.Horodecki et al. Phys. Lett. A **232** (1997), 333

Entanglement Measures	Results obtained from $\rho^{T_\mu}(t)$
1.PPT Criterion for an entanglement	Eigenvalues of $\rho^{T_\mu}(t)$ are $\lambda_1 = P_d$, $\lambda_2 = P_a$, $\lambda_3 = \sqrt{P_d P_a}$, and $\lambda_4 = -\sqrt{P_d P_a}$
2.Negativity [4] $N(\rho^{e\mu}) = \|\|\rho^{T_\mu}\|\| - 1$	$N_{e\mu} = 2\sqrt{P_a P_d}$

- Using the "Spin-flipped" density matrix, $\tilde{\rho}^{e\mu}(t) = (\sigma_y \otimes \sigma_y)\rho^{*e\mu}(t)(\sigma_y \otimes \sigma_y)$ where σ_x and σ_y are Pauli matrices, we calculate concurrence: $C(\rho^{e\mu}(t)) \equiv [max(\mu_1 - \mu_2 - \mu_3 - \mu_4, 0)]$, in which $\mu_1, ..., \mu_4$ are the eigenvalues of the matrix $\rho^{e\mu}(t)\tilde{\rho}^{e\mu}(t)$ and for a pure state $C\left|\psi_i\right\rangle = \left|\left\langle\psi\right|\sigma_y \otimes \sigma_y\left|\psi^*\right\rangle\right|$[5,6].
- Tangle: $\tau(\rho^{e\mu}) \equiv [max(\mu_1 - \mu_2 - \mu_3 - \mu_4, 0)]^2$ [7].
- Linear entropy: $S(\rho^{e\mu}) = 1 - Tr(\rho^{e\mu})^2$[2].

[4] Yong-Cheng. OU, et al.PhysRevA.75.062308 (2007)

[5] William K. Wooters, Phys.Rev.Lett.80.2245 (1998)

[6] A.K.Alok et al.Nucl. Phys. B **909** (2016)

[7] V.coffman et al.Phys. Rev. A 61, 052306 (2000)

Entanglement Measures	Results obtained from $\rho^{e\mu}(t)$
1.Concurrence $C(\rho^{e\mu}(t))$	Only one eigenvalue(μ) is non zero: $2\sqrt{P_a P_d}$ thus $C_{e\mu} = 2\sqrt{P_a P_d}$
2.Tangle $\tau(\rho^{e\mu})$	$\tau_{e\mu} = 4P_a P_d$
5. Linear Entropy $S(\rho^{e\mu})$	$S_{e\mu} = 4P_a P_d = \tau_{e\mu}$

- We see that entanglement measures between e and μ modes as [8]:

$$\tau_{e\mu} = C^2_{e\mu} = N^2_{e\mu} = S_{e\mu} = 4P_a P_d. \quad (4)$$

[8] A.K.Jha, arXiv:2004.14853v2 (2020) [hep-ph]

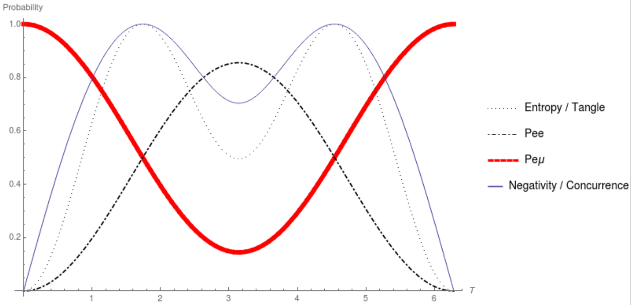

Figure: The time evolution of the various measures of entanglement compared to the oscillation probabilities in a typical reactor experiment. $\tau_{e\mu}$, $S_{e\mu}$ $N_{e\mu}$, $C_{e\mu}$, P_{ee}, and $P_{e\mu}$ as functions of the scaled time $T \equiv \Delta m^2 t/2E$ are plotted for $|\nu_e(t)\rangle$, considering the experimental value $\sin^2\theta = 0.310$, where $\theta=$ mixing angle [9].

- We see that when $P_{ee} = P_{e\mu} = 0.5$, all measures of entanglement tend to 1 i.e, $N_{e\mu} = \tau_{e\mu} = C_{e\mu} = S_{e\mu} = 1$, which corresponds to maximally entangled state.

[9] I. Esteban et al.JHEP **01** (2019), 106

Concurrence Plot

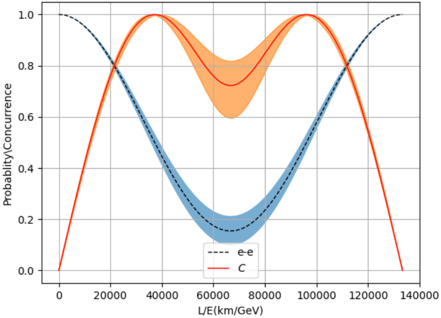

Fig.(a)

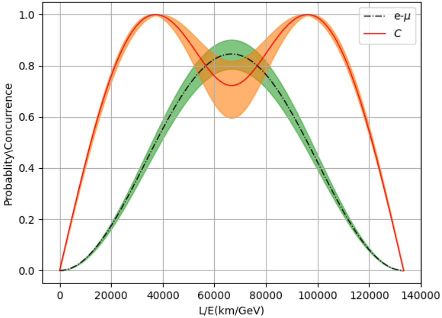

Fig.(b)

Figure: In Fig.(a), the blue band shows the ν_e disappearance probability (Black, dashed line) and the orange band shows the concurrence (Red, solid line) in two flavor neutrino oscillations [10]. Similarly, the green band in Fig.(b) shows the ν_e appearance probability (Black, dash dotted line).

[10] I. Esteban et al.JHEP **09** (2020), 178

Daya Bay

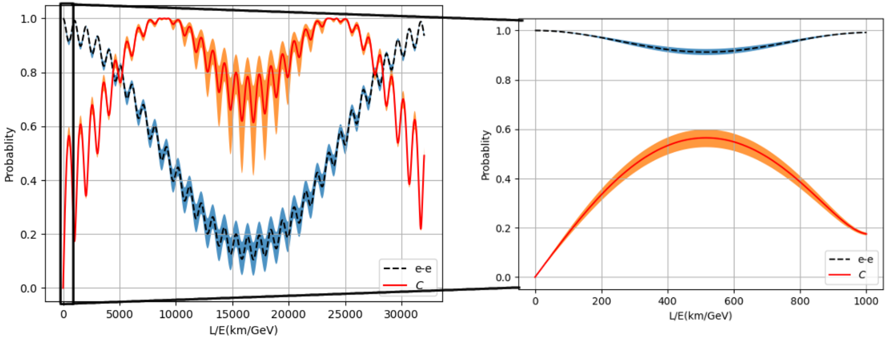

Figure: The blue band shows the short range ν_e disappearance probability (Black, dashed line) and the orange band shows concurrence (Red, solid line) in two flavor neutrino oscillations[11].

[11] F. P. An et al. (Daya Bay Collaboration), Phys. Rev. Lett. 115, 111802 (2015)

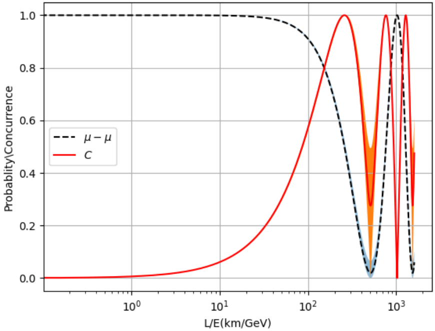

Figure: The blue band represents the long-range survival probability $\nu_\mu \rightarrow \nu_\mu$ (Black, dashed line) and it compared with orange band which gives concurrence (Red, solid line) in two flavor neutrino oscillations [12].

[12] A.B. Sousa (MINOS and MINOS+ Collaborations), First MINOS+data and new results from MINOS. AIP Conf. Proc. 1666, 110004 (2015)

Quantum Computer Circuit to simulate bi-partite entanglement in two flavor neutrino oscillations

- We identified that the SU(2) rotation matrix $R(\theta)$ can be encoded in the IBM quantum computer by using the universal quantum gate $U3$ [13,14]:

$$U3(\Phi, \psi, \lambda) = \begin{pmatrix} cos\frac{\Phi}{2} & -sin\frac{\Phi}{2}e^{i\lambda} \\ sin\frac{\Phi}{2}e^{i\psi} & cos\frac{\Phi}{2}e^{i(\lambda+\psi)} \end{pmatrix} \quad (5)$$

- In the two flavor neutrino oscillation, the parameters ψ and λ can be removed by rephasing the charged muon $|\nu_\mu\rangle$ field and $|\nu_2\rangle$ via

$$\begin{aligned} |\nu_\mu\rangle &\rightarrow e^{-i\psi}\,|\nu_\mu\rangle \\ |\nu_2\rangle &\rightarrow e^{i\lambda}\,|\nu_2\rangle. \end{aligned} \quad (6)$$

  Thus without loss of generality, we can set the parameter value $\psi = 0$ and $\lambda = 0$ in U3 gate.

[13] C.A. Argüelles et al.Phys.Rev.Research. 1 (2019) 033176

[14] G.B. Lesovik et al.Sci.Rep.9, 4396 (2019)

- For the two-neutrino system, oscillation probabilities are depend on one of the parameters of U3 gate. Thus,

$$R(\theta) = U3(2\theta, 0, 0) = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \equiv \begin{pmatrix} \tilde{U_{ee}} & \tilde{U_{e\mu}} \\ \tilde{U_{\mu e}} & \tilde{U_{\mu\mu}} \end{pmatrix} \quad (7)$$

- In quantum optics, the action of quantum mechanical beam splitter is given by $SU(2)$ rotation matrix $R(\theta)$ which performs exactly the same transformation on photons as the neutrino mixing matrix does.
- Thus, the action of $U3(2\theta, 0, 0)$ is defined as a beam splitter transformation and it can transfer each bit into qubit therefore constructing superposition states :

$$\begin{aligned} U3(2\theta, 0, 0)\,|0\rangle &\rightarrow \tilde{U}_{ee}\,|0\rangle + \tilde{U}_{e\mu}\,|1\rangle, \\ U3(2\theta, 0, 0)\,|1\rangle &\rightarrow \tilde{U}_{ee}\,|1\rangle + \tilde{U}_{e\mu}\,|0\rangle \end{aligned} \quad (8)$$

- The time-evolution operation can be identified as S-gate

$$S(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\psi} \end{pmatrix} = U1(t) \tag{9}$$

- For the two ν system,

$$\begin{pmatrix} |\nu_e(t)\rangle \\ |\nu_\mu(t)\rangle \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\psi} \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \tilde{U}_{ee}(t) & \tilde{U}_{e\mu}(t) \\ \tilde{U}_{\mu e}(t) & \tilde{U}_{\mu\mu}(t) \end{pmatrix} \begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \end{pmatrix} \tag{10}$$

where $\psi = \frac{\Delta m^2 t}{2E}$.

- Quantum cicuit embodying two-flavor neutrino oscillations contains following steps:
Prepare flavor state $\rightarrow$ Rotate to mass basis $\rightarrow$ Time evolution $\rightarrow$ Rotate from mass basis $\rightarrow$ Measures

- The gate arrangement of two flavor neutrino oscillations in one qubit mode system:

$$\begin{aligned} |\nu_e(t)\rangle &= U3(2\theta,0,0)U1(t)U3(-2\theta,0,0)\,|0\rangle \\ |\nu_\mu(t)\rangle &= U3(2\theta,0,0)U1(t)U3(-2\theta,0,0)X\,|0\rangle \end{aligned} \tag{11}$$

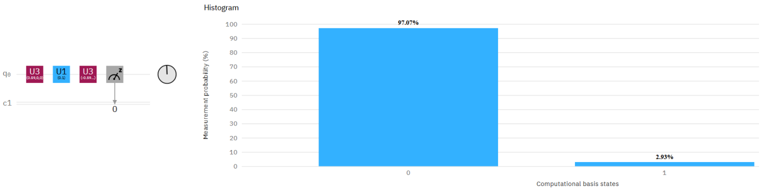

Figure: Two flavor neutrino oscillations encoded in one qubit mode system on IBMQ platform. Circuit diagram and Histogram plot (Transition probablities in percentage) for $P_{e\to e}$: $(|\nu_e(t)\rangle = \tilde{U}_{ee}(t)\,|0\rangle_e + \tilde{U}_{e\mu}(t)\,|1\rangle_\mu)$

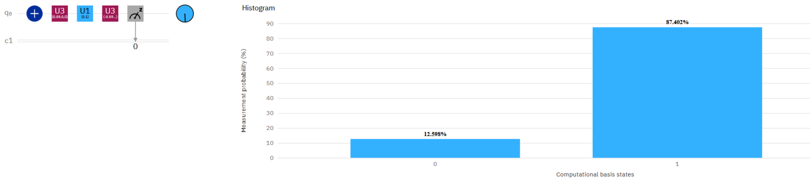

Figure: Two flavor neutrino oscillations encoded in one qubit mode system on IBMQ platform. Circuit diagram and Histogram plot (Transition probablities in percentage) for $P_{\mu\rightarrow e}$: $|\nu_\mu(t)\rangle = \tilde{U}_{e\mu}(t)\,|0\rangle_e + \tilde{U}_{\mu\mu}(t)\,|1\rangle_\mu$

- In the two-qubit system we first prepare a quantum circuit of bi-partite electron neutrino state in the linear superposition of mass mode basis:

$$|\nu_e(0)\rangle = CNOT_{12}(U3(-2\theta, 0, 0)\,|0\rangle_1 \otimes X\,|0\rangle_2) \quad (12)$$

$$\rightarrow CNOT_{12}(\tilde{U}_{ee}\,|1\rangle_1 + \tilde{U}_{e\mu}\,|0\rangle_1) \otimes |1\rangle_2$$
$$\rightarrow \tilde{U}_{ee}(t)\,|10\rangle_1 + \tilde{U}_{e\mu}(t)\,|01\rangle_2$$

- The $CNOT_{12}$ gate is defined as if the control qubit (first (1) qubit) is in the state $|0\rangle$ the target qubit (second (2) qubit) is not affected, conversely if the control qubit in the state $|1\rangle$, the target is flipped

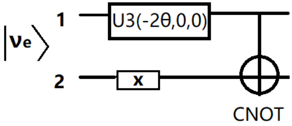

Figure: The circuit represent an electron-flavor neutrino state in a linear superposition of mass mode basis in two qubit system: $|\nu_e\rangle = \tilde{U}_{ee}\,|10\rangle_1 + \tilde{U}_{e\mu}\,|01\rangle_2$. Here, 1 and 2 represent the input qubits first and second, respectively.

- The gate arrangement of the time evolved electron flavor neutrino state in the two qubit flavor mode system:

$$
\begin{aligned}
|\nu_e(t)\rangle = CNOT_{12}(U3(2\theta,0,0)U1(t) \\
U3(-2\theta,0,0)\,|0\rangle_1 \otimes X\,|0\rangle_2) \\
\rightarrow \tilde{U}_{ee}(t)\,|10\rangle_e + \tilde{U}_{e\mu}(t)\,|01\rangle_\mu
\end{aligned}
\tag{13}
$$

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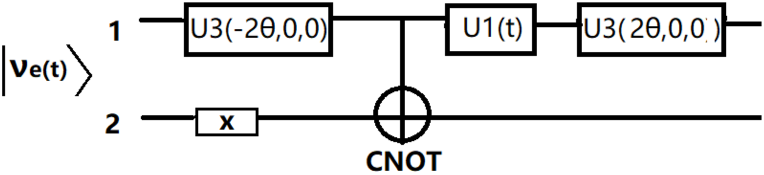


Figure: The circuit represent the time evolved electron flavor neutrino state in a linear superposition of flavor basis (ν_e disappearance), in the two qubit system: $|\nu_e(t)\rangle = \tilde{U}_{ee}(t)\,|10\rangle_e + \tilde{U}_{e\mu}(t)\,|01\rangle_\mu$.

- The gate arrangement of the time evolved muon flavor neutrino state in the two qubit mode system:

$$|\nu_\mu(t)\rangle = CNOT_{12}(U3(2\theta,0,0)U1(t) \quad (14)$$
$$U3(-2\theta,0,0)X|0\rangle_1 \otimes X|0\rangle_2)$$
$$\rightarrow \tilde{U}_{\mu e}(t)|10\rangle_e + \tilde{U}_{\mu\mu}(t)|01\rangle_\mu \quad (15)$$

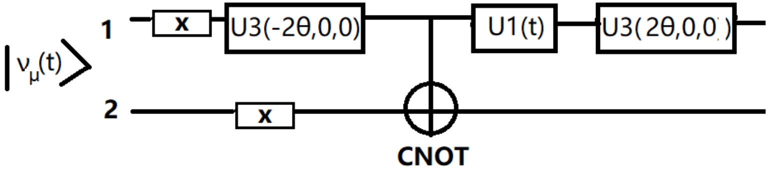


Figure: The circuit represent the time evolved muon flavor neutrino state in a linear superposition of flavor basis ($\nu_\mu \rightarrow \nu_e$), in the two qubit system: $|\nu_\mu(t)\rangle = \tilde{U}_{\mu e}(t)|10\rangle_e + \tilde{U}_{\mu\mu}(t)|01\rangle_\mu$.

Direct measurement of concurrence for the two neutrino system on a quantum computer

- First we prepare two decouple copies of bi-partite neutrino state $|\nu_\alpha(t)\rangle \otimes |\nu_\alpha(t)\rangle$ in the two flavor system (where $\alpha = e, \mu$), and apply a "spin-flipped" operation $\sigma_y \otimes \sigma_y$ on one of the two copies to prepare an arbitrary global state of neutrino in the four qubit system.
- We find that the concurrence value of the time evolved flavor neutrino oscillation can be extracted from the global state [15].

[15] G.Romero et al.Phys.Rev.A.75.032303 (2007)

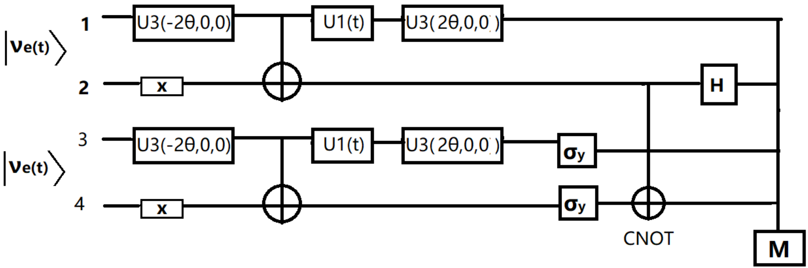


Figure: The circuit represent the concurrence measurement of ν_e disappearance in two-flavor neutrino oscillations.

- The first two channels (1 and 2) stand for the entangled state $|\nu_e(t)\rangle$ that we want to measure. The third and fourth channel (3 and 4) denote the copy of $|\nu_e(t)\rangle$

- Take two copies of time evolved electron flavor state ($\nu_e(t)$) $|\nu_e(t)\rangle \otimes |\nu_e(t)\rangle$, and apply "spin-flipped" operation $\sigma_y \otimes \sigma_y$ on the second copy such that the global state is described by

$$
\begin{aligned}
|\Phi(t)\rangle &= |\nu_e(t)\rangle \otimes (\sigma_y \otimes \sigma_y |\nu_e(t)\rangle) \\
&= (\tilde{U}_{ee}(t)|10\rangle + \tilde{U}_{e\mu}(t)|01\rangle) \otimes (\tilde{U}_{ee}(t)|01\rangle + \tilde{U}_{e\mu}(t)|10\rangle) \\
&= |\tilde{U}_{ee}(t)|^2 |1001\rangle + \tilde{U}_{ee}(t)\tilde{U}_{e\mu}|1010\rangle + \tilde{U}_{e\mu}(t)\tilde{U}_{ee}|0101\rangle \\
&+ |\tilde{U}_{ee}(t)|^2 |0110\rangle. \qquad (16)
\end{aligned}
$$

- Now let us apply $CNOT_{24}$ operation between second (2) and fourth (4) qubit, and the target qubit (4) is inverted only when the control qubit (2) is $|1\rangle$ i.e, $|0101\rangle \rightarrow |0100\rangle$ and $|0110\rangle \rightarrow |0111\rangle$, such that

$$
\begin{aligned}
|\Phi_1(t)\rangle &= |\tilde{U}_{ee}(t)|^2 |1001\rangle + \tilde{U}_{ee}(t)\tilde{U}_{e\mu}|1010\rangle \\
&+ \tilde{U}_{e\mu}(t)\tilde{U}_{ee}|0100\rangle + |\tilde{U}_{ee}(t)|^2 |0111\rangle. \qquad (17)
\end{aligned}
$$

- Finally, we perform the Hadamard transformation, $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on the second (2) qubit. The H operation can transfer each qubit as:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \quad (18)$$

- Hence, the state of the overall system become

$$|\Phi_2(t)\rangle = \frac{1}{\sqrt{2}}[|\tilde{U}_{ee}(t)|^2|1001\rangle - |\tilde{U}_{ee}(t)|^2|1101\rangle + \tilde{U}_{ee}(t)\tilde{U}_{e\mu}(t) \\ |1010\rangle - \tilde{U}_{ee}(t)\tilde{U}_{e\mu}(t)|1110\rangle + \tilde{U}_{e\mu}(t)\tilde{U}_{ee}(t)|0100\rangle \\ +\tilde{U}_{e\mu}(t)\tilde{U}_{ee}(t)|0000\rangle + |\tilde{U}_{e\mu}(t)|^2|0111\rangle + |\tilde{U}_{e\mu}(t)|^2|0011\rangle] \quad (19)$$

- Thus, we observe from eq.(19) that the concurrence information of the electron neutrino flavor state $|\nu_e(t)\rangle$ is present in the coefficient $\tilde{U}_{e\mu}(t)\tilde{U}_{ee}(t)$ through

$$C(|\nu_e(t)\rangle) = 2\sqrt{2P_{0000}} = 2\sqrt{P_a P_d}, \quad (20)$$

where $P_{0000} = \frac{|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\mu}(t)|^2}{2} = \frac{P_a P_d}{2}$.

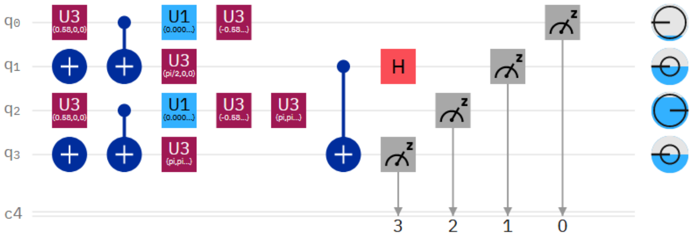


Figure: Concurrence circuit for the two qubit ν_e disappearance bi-partite state on the IBMQ platform

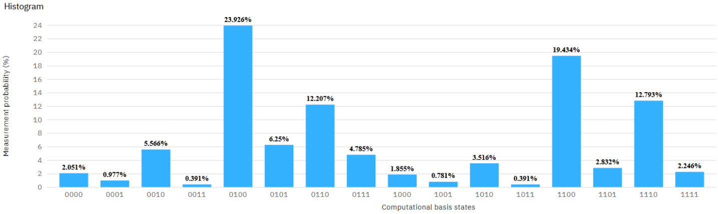
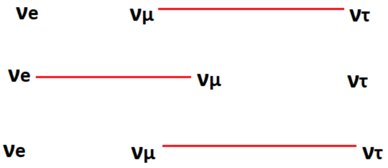


Figure: The concurrence varies with time at the IBMQ computer for an initial electron neutrino flavor state . The concurrence information encoded in the coefficients of four qubit global state basis are shown through Histogram (probabilities in percentage) plot.

Tri-Partite Entanglement In Three-Flavor Neutrino Oscillations

- Biseparable states are formed in a three particle system, by considering two out of three modes state as a single state.



Bi-partite Entanglement in Tri-partite System (Biseparable)

- The density matrix of the time evolved electron neutrino flavor state is $\rho_{e\mu\tau}(t) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\tilde{U}_{ee}(t)|^2 & 0 & \tilde{U}_{ee}(t)\tilde{U}^*_{e\mu}(t) & \tilde{U}_{ee}(t)\tilde{U}^*_{e\tau}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{U}_{e\mu}(t)\tilde{U}^*_{ee}(t) & 0 & |\tilde{U}_{e\mu}(t)|^2 & \tilde{U}_{e\mu}(t)\tilde{U}^*_{e\tau}(t) & 0 \\ 0 & 0 & 0 & \tilde{U}_{e\tau}(t)\tilde{U}^*_{ee}(t) & 0 & \tilde{U}_{e\tau}(t)\tilde{U}^*_{e\mu}(t) & |\tilde{U}_{e\tau}(t)|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The pairwise measures of entanglement are negativity ($N^2_{e(\mu\tau)}$), Concurrence ($C^2_{e(\mu\tau)}$), tangle ($\tau_{e(\mu\tau)}$) and Linear entropy ($S_{e(\mu\tau)}$). Because we have effectively reduced the three ν system to bi-partite system, the measures remain the same.

$$N^2_{e(\mu\tau)} = C^2_{e(\mu\tau)} = \tau_{e(\mu\tau)} = S_{e(\mu\tau)} = 4P_aP_d \tag{21}$$

- For tri-partite entanglement, criterion is known as **Coffman-Kundu-Wooters (CKW) inequality**. It states that the sum of quantum correlations between e and μ, and between e and τ, is either less than or equal to the quantum correlations between e and $\mu\tau$ (treating it as a single object) [4,16]: $C^2_{e\mu} + C^2_{e\tau} \leq C^2_{e(\mu\tau)}$, $\tau_{e\mu} + \tau_{e\tau} \leq \tau_{e(\mu\tau)}$ and $N^2_{e\mu} + N^2_{e\tau} \leq N^2_{e(\mu\tau)}$.

Bi-separable entanglement measures	Results from $\rho^{e\mu\tau}(t)$
1. Concurrence equality	$C^2_{e\mu} + C^2_{e\tau} = C^2_{e(\mu\tau)}$
2. Tangle equality	$\tau_{e\mu} + \tau_{e\tau} = \tau_{e(\mu\tau)}$
3. Negativity inequality	$N^2_{e\mu} + N^2_{e\tau} < N^2_{e(\mu\tau)}$

[16] V.Coffman et al. Phys.Rev.A.61.052306,(2000)

- There are two extra measure for genuine tri-partite entanglement quantified by **three-tangle** and **three-**π negativity known as residual entanglement.
- The residual entanglement three-π for electron neutrino flavor state $|\nu_e(t)\rangle$ is,

$$
\begin{aligned}
\pi_{e\mu\tau} &= \frac{4}{3}[|\tilde{U}_{ee}(t)|^2\sqrt{|\tilde{U}_{ee}(t)|^4 + 4|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\tau}(t)|^2} \\
&+ |\tilde{U}_{e\mu}(t)|^2\sqrt{|\tilde{U}_{e\mu}(t)|^4 + 4|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\tau}(t)|^2} \\
&+ |\tilde{U}_{e\tau}(t)|^2\sqrt{|\tilde{U}_{e\tau}(t)|^4 + 4|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\mu}(t)|^2} \\
&- |\tilde{U}_{ee}(t)|^4 - |\tilde{U}_{e\mu}(t)|^4 - |\tilde{U}_{e\tau}(t)|^4]. \qquad (22)
\end{aligned}
$$

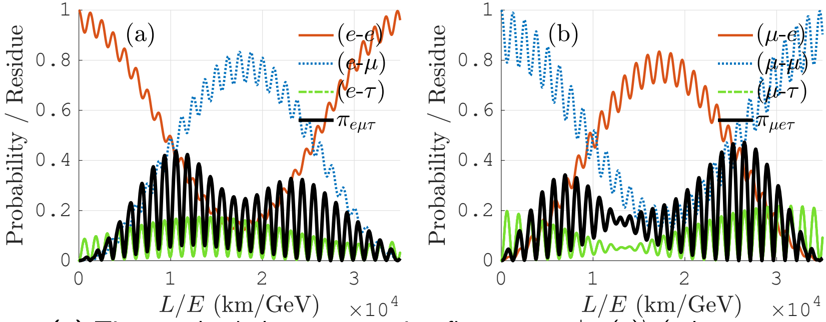
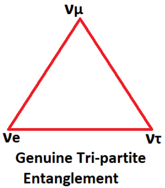


Figure: **(a).**Time evolved electron neutrino flavor state $|\nu_e(t)\rangle$ (relevant to reactor experiment) and **(b)** a muon flavor state $|\nu_\mu(t)\rangle$ (relevant to accelerator experiment) vs scale of distance per energy unit $\frac{L}{E}$. At $\frac{L}{E} > 0$ entanglement among three-flavor modes occurs i.e, the black curve $\pi_{e\mu\tau} > 0$ or $\pi_{\mu e\tau} > 0$, and exhibits a typical oscillatory behavior. $\pi_{e\mu\tau}$ reaches the maximum value 0.436629 (see Fig.2(a)) when transition probabilities are $P_{\nu_{e\to e}} = 0.39602$, $P_{\nu_{e\to\mu}} = 0.435899$, and $P_{\nu_{e\to\tau}} = 0.168081$. Similarly, for $|\nu_\mu(t)\rangle$, $\pi_{\mu e\tau}$ reaches the maximum value 0.472629 (see Fig.2(b)) indicating genuine tri-partite entanglement.[17]

[17] I.Esteban et al.JHEP 01 (2019), 106

Residual Entanglement	Tri-Partite results for ν_e disappearance
Three-tangle $\tau_{e\mu\tau} = C^2_{e(\mu\tau)} - C^2_{e\mu} + C^2_{e\tau}$	$\tau_{e\mu\tau} = 0$
Three-π $\pi_{e\mu\tau} = \frac{1}{3}(N^2_{e(\mu\tau)} + N^2_{\mu(e\tau)} + N^2_{\tau(e\mu)} - 2N^2_{e\mu} - 2N^2_{e\tau} - 2N^2_{\mu\tau})$	$\pi_{e\mu\tau} > 0$

- The residual entanglement three-tangle vanish, but the non-zero value of three-π result shows that the three flavor neutrino oscillations has a genuine form of three way entanglement [18].
- Correlations exhibited by neutrino oscillations in tri-partite system are like the W-states which are legitimate physical resources for quantum information tasks.



[18] A.K.Jha, arXiv:2004.14853v2 (2020) [hep-ph]

Summary

- We mapped the flavor state to two and three mode states which are like qubits and shows the quantification of bipartite measures like concurrence, tangle, negativity and linear entropy in two and three-flavor neutrino oscillations.
- The concurrence plot has been shown for the short range $\nu_e \rightarrow \nu_e$ and long range $\nu_\mu \rightarrow \nu_\mu$ survival probabilities in the two neutrino system.
- We constructed quantum computer circuit using Universal $U(3)$ gate, S-gate, Controlled-NOT and Pauli (X) gate to outline the simulation of two flavor neutrino oscillations on a quantum computer.
- We directly measure the concurrence using spin-flipped $\sigma_y \otimes \sigma_y$ gate, and Hadamard gate, and proposed the implications of the implementation of entanglement in the two neutrino system on the IBM quantum processor.
- The tri-partite result $\pi_{e\mu\tau} > \tau_{e\mu\tau} = 0$ or $\pi_{\mu e\tau} > \tau_{\mu e\tau} = 0$ imply that the three-neutrino state shows the remarkable property of having a genuine form of three way entanglement. Work on simulation of such systems on a quantum computer is in progress.

Thank You