Generic algorithm for multi-particle cumulants of azimuthal correlations at the LHC

The 5th international conference on particle physics and astrophysics

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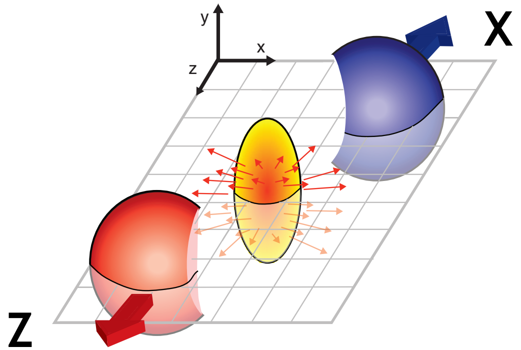
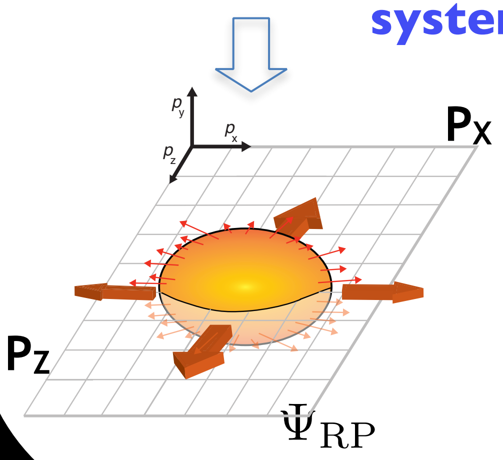
VILLUM FONDEN ✕ VELUX FONDEN

Anisotropic Flow

- Spatial anisotropy in the initial state converted to momentum anisotropic particle distributions
  - known as ***anisotropic flow***
  - reflect initial **anisotropy** and **transport properties** of QGP



system expansion



Initial state

$$\varepsilon_n = \frac{\sqrt{\langle r^n \cos(n\phi)\rangle^2 + \langle r^n \sin(n\phi)\rangle^2}}{\langle r^n\rangle}$$

Initial spatial Anisotropy

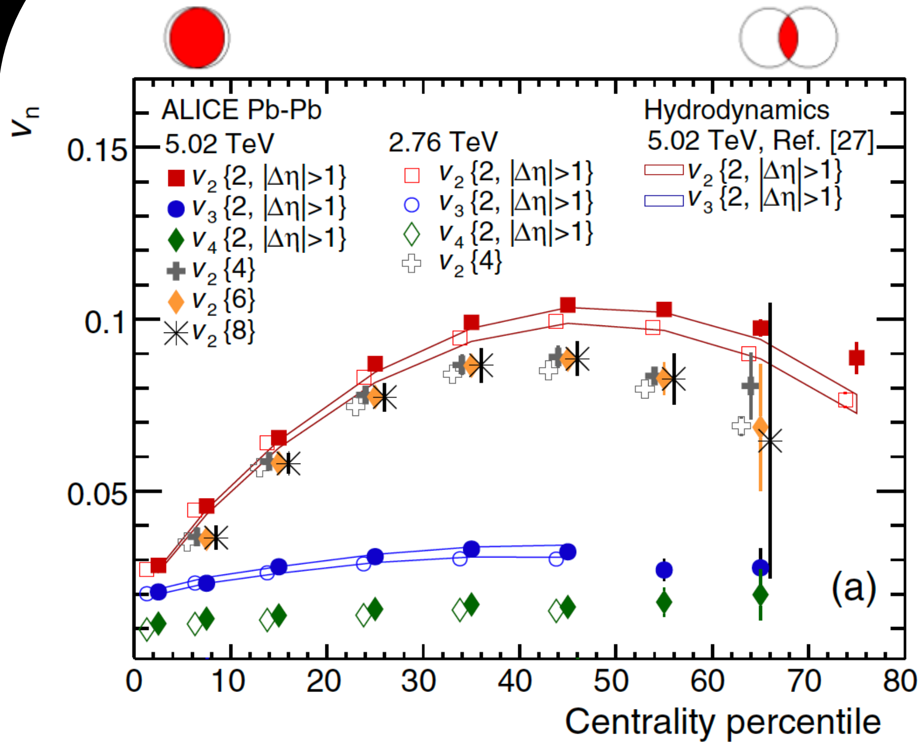
Final state

$$v_n = \langle \cos n(\varphi - \Psi_n)\rangle$$

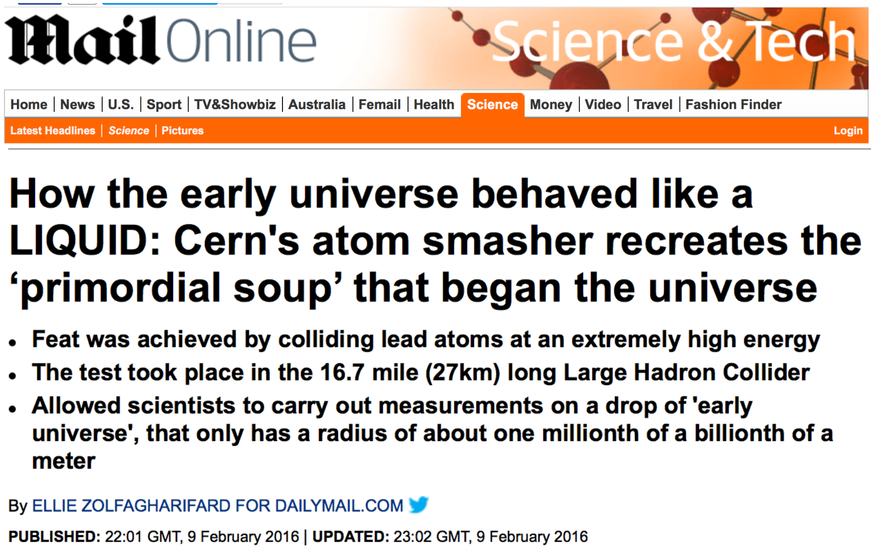
momentum space Anisotropic Flow

Flowing primordial fluid

ALICE, Physical Review Letters 117, 182301 (2016)



Daily Mail, February 2016



How the early universe behaved like a LIQUID: Cern's atom smasher recreates the 'primordial soup' that began the universe

- **Feat was achieved by colliding lead atoms at an extremely high energy**
- **The test took place in the 16.7 mile (27km) long Large Hadron Collider**
- **Allowed scientists to carry out measurements on a drop of 'early universe', that only has a radius of about one millionth of a billionth of a meter**

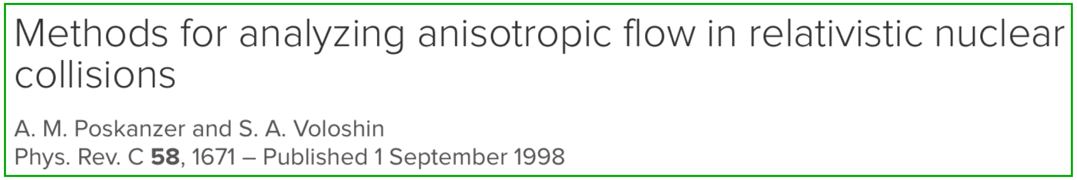
By ELLIE ZOLFAGHARIFARD FOR DAILYMAIL.COM

PUBLISHED: 22:01 GMT, 9 February 2016 | **UPDATED:** 23:02 GMT, 9 February 2016

- Anisotropic flow measurements agree with hydrodynamics
    - The Quark-Gluon Plasma produced at the top LHC energy behaves like a perfect fluid

How to measure flow

❖ Event-Plane method, popular at RHIC



Methods for analyzing anisotropic flow in relativistic nuclear collisions

A. M. Poskanzer and S. A. Voloshin
Phys. Rev. C **58**, 1671 – Published 1 September 1998

1235 citations

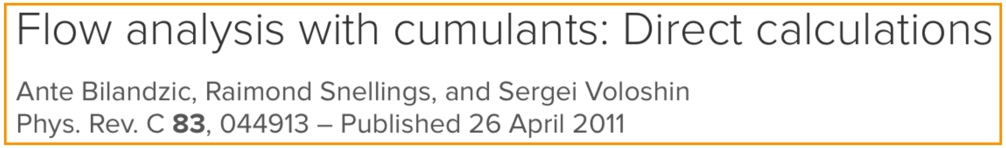
❖ Multi-particle correlations with generating function method, popular at RHIC

Flow analysis from multiparticle azimuthal correlations

Nicolas Borghini, Phuong Mai Dinh, and Jean-Yves Ollitrault
Phys. Rev. C **64**, 054901 – Published 25 September 2001

408 citations

❖ Multi-particle correlations with Q-cumulant method, popular at earlier LHC



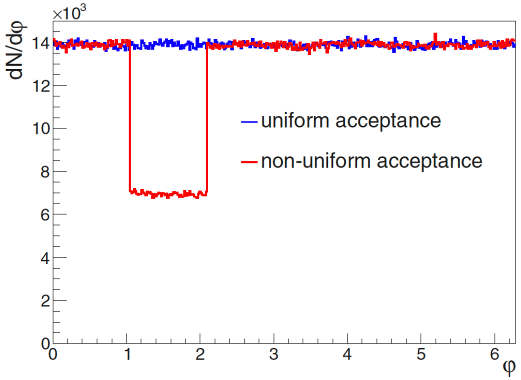
Flow analysis with cumulants: Direct calculations

Ante Bilandzic, Raimond Snellings, and Sergei Voloshin
Phys. Rev. C **83**, 044913 – Published 26 April 2011

317 citations

Issues of Q-cumulant method

- Unable to provide precise corrections on non-uniform acceptance (NUA) and efficiency (NUE) corrections
  - biased flow measurements



- Complication of the design of the code for m-particle correlations
  - For 4-particle correlations of

$\langle \cos n(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4) \rangle$ for $v_n\{4\}$ calculations

$\langle \cos n(3\varphi_1 + 2\varphi_2 - 3\varphi_3 - 2\varphi_4) \rangle$ for Symmetric cumulants SC(3,2)

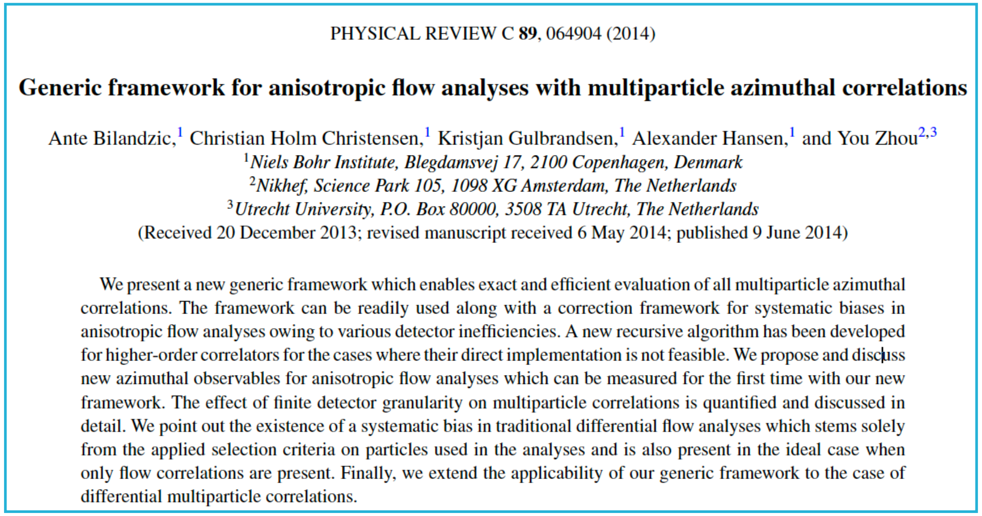
$$\begin{aligned}
\langle 4 \rangle_{n,n|n,n} &\equiv \langle \cos(n(\phi_1+\phi_2-\phi_3-\phi_4)) \rangle \\
&= \frac{1}{\binom{M}{4}4!} \sum_{\substack{i,j,k,l=1 \\ (i \neq j \neq k \neq l)}}^{M} e^{in(\phi_i+\phi_j-\phi_k-\phi_l)} \\
&= \frac{1}{\binom{M}{4}4!} \times \Big[|Q_n|^4 + |Q_{2n}|^2 - 2 \cdot \mathfrak{Re}\left[Q_{2n} Q_n^* Q_n^*\right] - 4(M-2)\,|Q_n|^2 \\
&\quad +2M(M-3) \Big] \,.
\end{aligned}$$

$$\begin{aligned}
\langle 4 \rangle_{3n,2n|3n,2n} &\equiv \langle \cos(n(3\phi_1+2\phi_2-3\phi_3-2\phi_4)) \rangle \\
&= \frac{1}{\binom{M}{4}4!} \sum_{\substack{i,j,k,l=1 \\ (i \neq j \neq k \neq l)}}^{M} e^{in(3\phi_i+2\phi_j-3\phi_k-2\phi_l)} \\
&= \frac{1}{\binom{M}{4}4!} \times \Big[|Q_{3n}|^2 |Q_{2n}|^2 - 2 \cdot Q_{5n} Q_{3n}^* Q_{2n}^* - 2 \cdot Q_{3n} Q_{2n}^* Q_n^* \\
&\quad + |Q_{5n}|^2 - (M-4)(|Q_{3n}|^2 + |Q_{2n}|^2) + M(M-6) \Big] \,.
\end{aligned}$$

Completely different equations, different codes

Generic Framework

❖ 2014, Generic framework method, the most popular approach at the LHC



PHYSICAL REVIEW C **89**, 064904 (2014)

Generic framework for anisotropic flow analyses with multiparticle azimuthal correlations

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(Received 20 December 2013; revised manuscript received 6 May 2014; published 9 June 2014)

We present a new generic framework which enables exact and efficient evaluation of all multiparticle azimuthal correlations. The framework can be readily used along with a correction framework for systematic biases in anisotropic flow analyses owing to various detector inefficiencies. A new recursive algorithm has been developed for higher-order correlators for the cases where their direct implementation is not feasible. We propose and discuss new azimuthal observables for anisotropic flow analyses which can be measured for the first time with our new framework. The effect of finite detector granularity on multiparticle correlations is quantified and discussed in detail. We point out the existence of a systematic bias in traditional differential flow analyses which stems solely from the applied selection criteria on particles used in the analyses and is also present in the ideal case when only flow correlations are present. Finally, we extend the applicability of our generic framework to the case of differential multiparticle correlations.

154 citations

Step1 → Step2 → Step3

$$Q_{n,p} \equiv \sum_{k=1}^{M} w_k^{p}\, e^{in\varphi_k}$$

$$\begin{aligned}\mathrm{N}\langle m\rangle_{n_1,n_2,\ldots,n_m} &\equiv \sum_{\substack{k_1,k_2,\ldots,k_m=1\\ k_1\neq k_2\neq\cdots\neq k_m}}^{M} w_{k_1} w_{k_2}\cdots w_{k_m} \\ &\quad\times e^{i(n_1\varphi_{k_1}+n_2\varphi_{k_2}+\cdots+n_m\varphi_{k_m})},\\ \mathrm{D}\langle m\rangle_{n_1,n_2,\ldots,n_m} &\equiv \sum_{\substack{k_1,k_2,\ldots,k_m=1\\ k_1\neq k_2\neq\cdots\neq k_m}}^{M} w_{k_1} w_{k_2}\cdots w_{k_m}\\ &= \mathrm{N}\langle m\rangle_{0,0,\ldots,0}.\end{aligned}$$

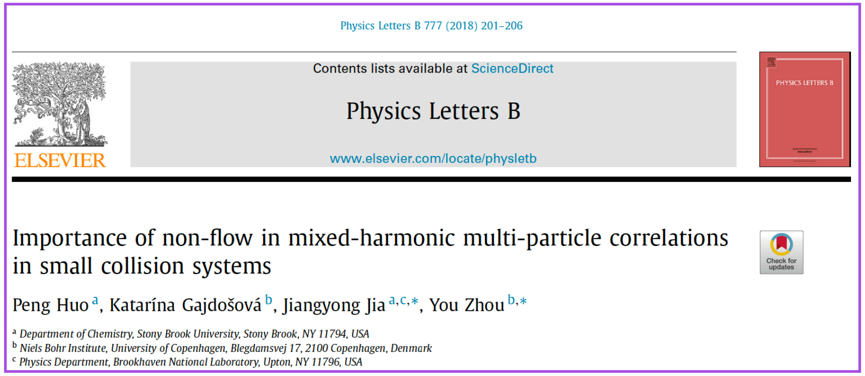
$$\langle m\rangle_{n_1,n_2,\ldots,n_m} \equiv \left\langle e^{i(n_1\varphi_{k_1}+n_2\varphi_{k_2}+\cdots+n_m\varphi_{k_m})}\right\rangle \equiv \frac{\sum\limits_{\substack{k_1,k_2,\ldots,k_m=1\\ k_1\neq k_2\neq\cdots\neq k_m}}^{M} w_{k_1} w_{k_2}\cdots w_{k_m}\, e^{i(n_1\varphi_{k_1}+n_2\varphi_{k_2}+\cdots+n_m\varphi_{k_m})}}{\sum\limits_{\substack{k_1,k_2,\ldots,k_m=1\\ k_1\neq k_2\neq\cdots\neq k_m}}^{M} w_{k_1} w_{k_2}\cdots w_{k_m}}.$$

Advantages of Generic Framework

- Clear advantage compared to Q-cumulant

$$
\begin{aligned}
\mathrm{N}\langle 4\rangle_{n_1,n_2,n_3,n_4} = {} & Q_{n_1,1}Q_{n_2,1}Q_{n_3,1}Q_{n_4,1} - Q_{n_1+n_2,2}Q_{n_3,1}Q_{n_4,1} - Q_{n_2,1}Q_{n_1+n_3,2}Q_{n_4,1} \\
& - Q_{n_1,1}Q_{n_2+n_3,2}Q_{n_4,1} + 2Q_{n_1+n_2+n_3,3}Q_{n_4,1} - Q_{n_2,1}Q_{n_3,1}Q_{n_1+n_4,2} \\
& + Q_{n_2+n_3,2}Q_{n_1+n_4,2} - Q_{n_1,1}Q_{n_3,1}Q_{n_2+n_4,2} + Q_{n_1+n_3,2}Q_{n_2+n_4,2} \\
& + 2Q_{n_3,1}Q_{n_1+n_2+n_4,3} - Q_{n_1,1}Q_{n_2,1}Q_{n_3+n_4,2} + Q_{n_1+n_2,2}Q_{n_3+n_4,2} \\
& + 2Q_{n_2,1}Q_{n_1+n_3+n_4,3} + 2Q_{n_1,1}Q_{n_2+n_3+n_4,3} - 6Q_{n_1+n_2+n_3+n_4,4}, \\
\mathrm{D}\langle 4\rangle_{n_1,n_2,n_3,n_4} = {} & \mathrm{N}\langle 4\rangle_{0,0,0,0} = Q_{0,1}^4 - 6Q_{0,1}^2 Q_{0,2} + 3Q_{0,2}^2 + 8Q_{0,1}Q_{0,3} - 6Q_{0,4}.
\end{aligned}
$$

  - The same code for all multi-particle correlations
  - Take care of acceptance and efficiency corrections
- Further implementation with sub-event method
  - Largely suppress non-flow in multi-particle cumulants



Physics Letters B 777 (2018) 201–206

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Importance of non-flow in mixed-harmonic multi-particle correlations in small collision systems

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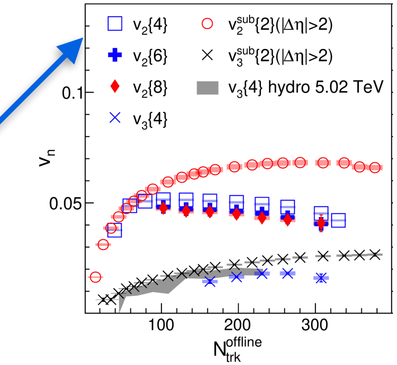
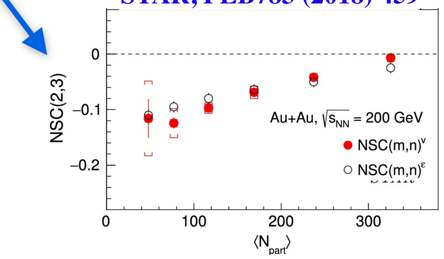
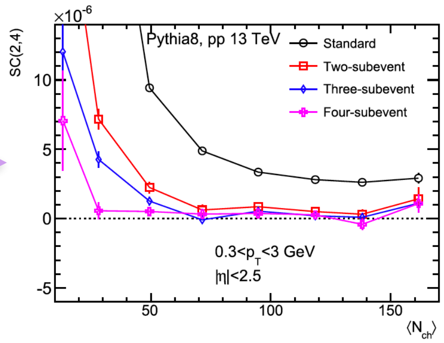
CMS, PRC101 (2020) 014912



STAR, PLB783 (2018) 459



P. Huo et al., PL B 777 (2018) 201

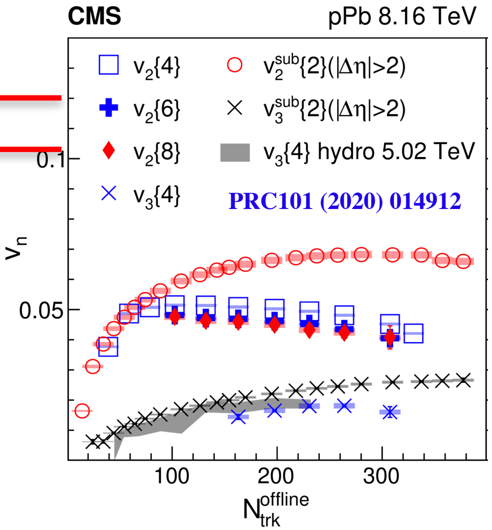


Disadvantages

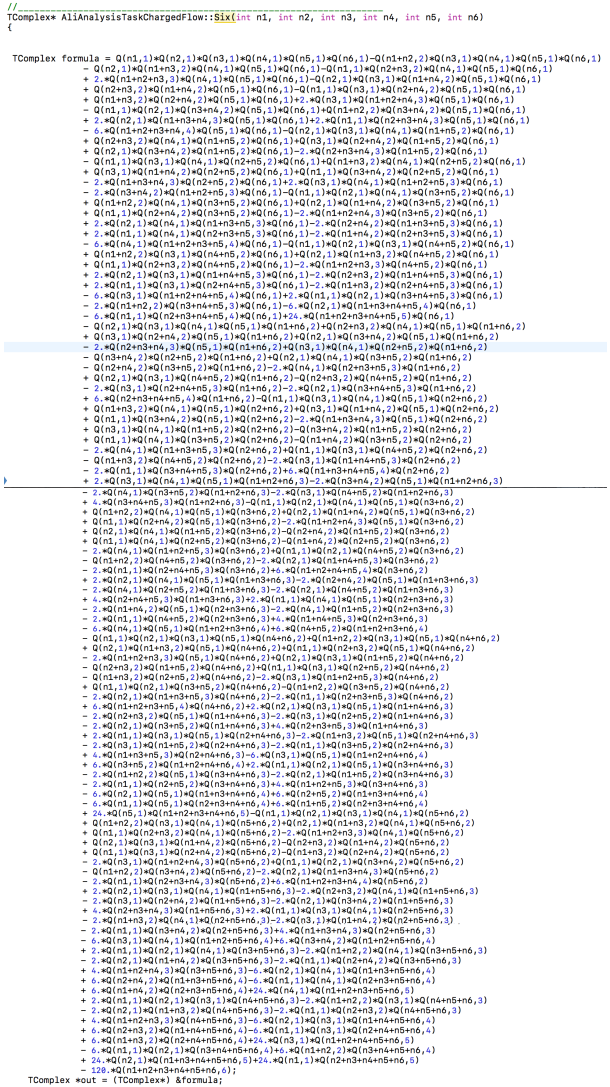
- **Disadvantage**
  - Terms of particle correlations following Bell numbers
  - For 6-particle correlations, there are **203** terms (see RHS)
  - For 8- and 10-particle correlations, there are ***4140*** and ***115975*** terms, respectively.
  - Impossible to implement (in C++)

6-particles

8-particles



Example: 6-particle correlations



```
//________________________________________________________________
TComplex* AliAnalysisTaskChargedFlow::Six(int n1, int n2, int n3, int n4, int n5, int n6)
{


 TComplex formula = Q(n1,1)*Q(n2,1)*Q(n3,1)*Q(n4,1)*Q(n5,1)*Q(n6,1)-Q(n1+n2,2)*Q(n3,1)*Q(n4,1)*Q(n5,1)*Q(n6,1)
              - Q(n2,1)*Q(n1+n3,2)*Q(n4,1)*Q(n5,1)*Q(n6,1)-Q(n1,1)*Q(n2+n3,2)*Q(n4,1)*Q(n5,1)*Q(n6,1)
              + 2.*Q(n1+n2+n3,3)*Q(n4,1)*Q(n5,1)*Q(n6,1)-Q(n2,1)*Q(n3,1)*Q(n1+n4,2)*Q(n5,1)*Q(n6,1)
              + Q(n2+n3,2)*Q(n1+n4,2)*Q(n5,1)*Q(n6,1)-Q(n1,1)*Q(n3,1)*Q(n2+n4,2)*Q(n5,1)*Q(n6,1)
              + Q(n1+n3,2)*Q(n2+n4,2)*Q(n5,1)*Q(n6,1)+2.*Q(n3,1)*Q(n1+n2+n4,3)*Q(n5,1)*Q(n6,1)
              - Q(n1,1)*Q(n2,1)*Q(n3+n4,2)*Q(n5,1)*Q(n6,1)+Q(n1+n2,2)*Q(n3+n4,2)*Q(n5,1)*Q(n6,1)
              + 2.*Q(n2,1)*Q(n1+n3+n4,3)*Q(n5,1)*Q(n6,1)+2.*Q(n1,1)*Q(n2+n3+n4,3)*Q(n5,1)*Q(n6,1)
              - 6.*Q(n1+n2+n3+n4,4)*Q(n5,1)*Q(n6,1)-Q(n2,1)*Q(n3,1)*Q(n4,1)*Q(n1+n5,2)*Q(n6,1)
              + Q(n2+n3,2)*Q(n4,1)*Q(n1+n5,2)*Q(n6,1)+Q(n3,1)*Q(n2+n4,2)*Q(n1+n5,2)*Q(n6,1)
              + Q(n2,1)*Q(n3+n4,2)*Q(n1+n5,2)*Q(n6,1)-2.*Q(n2+n3+n4,3)*Q(n1+n5,2)*Q(n6,1)
              - Q(n1,1)*Q(n3,1)*Q(n4,1)*Q(n2+n5,2)*Q(n6,1)+Q(n1+n3,2)*Q(n4,1)*Q(n2+n5,2)*Q(n6,1)
              + Q(n3,1)*Q(n1+n4,2)*Q(n2+n5,2)*Q(n6,1)+Q(n1,1)*Q(n3+n4,2)*Q(n2+n5,2)*Q(n6,1)
              - 2.*Q(n1+n3+n4,3)*Q(n2+n5,2)*Q(n6,1)+2.*Q(n3,1)*Q(n4,1)*Q(n1+n2+n5,3)*Q(n6,1)
              - 2.*Q(n3+n4,2)*Q(n1+n2+n5,3)*Q(n6,1)-Q(n1,1)*Q(n2,1)*Q(n4,1)*Q(n3+n5,2)*Q(n6,1)
              + Q(n1+n2,2)*Q(n4,1)*Q(n3+n5,2)*Q(n6,1)+Q(n2,1)*Q(n1+n4,2)*Q(n3+n5,2)*Q(n6,1)
              + Q(n1,1)*Q(n2+n4,2)*Q(n3+n5,2)*Q(n6,1)-2.*Q(n1+n2+n4,3)*Q(n3+n5,2)*Q(n6,1)
              + 2.*Q(n2,1)*Q(n4,1)*Q(n1+n3+n5,3)*Q(n6,1)-2.*Q(n2+n4,2)*Q(n1+n3+n5,3)*Q(n6,1)
              + 2.*Q(n1,1)*Q(n4,1)*Q(n2+n3+n5,3)*Q(n6,1)-2.*Q(n1+n4,2)*Q(n2+n3+n5,3)*Q(n6,1)
              - 6.*Q(n4,1)*Q(n1+n2+n3+n5,4)*Q(n6,1)-Q(n1,1)*Q(n2,1)*Q(n3,1)*Q(n4+n5,2)*Q(n6,1)
              + Q(n1+n2,2)*Q(n3,1)*Q(n4+n5,2)*Q(n6,1)+Q(n2,1)*Q(n1+n3,2)*Q(n4+n5,2)*Q(n6,1)
              + Q(n1,1)*Q(n2+n3,2)*Q(n4+n5,2)*Q(n6,1)-2.*Q(n1+n2+n3,3)*Q(n4+n5,2)*Q(n6,1)
              + 2.*Q(n2,1)*Q(n3,1)*Q(n1+n4+n5,3)*Q(n6,1)-2.*Q(n2+n3,2)*Q(n1+n4+n5,3)*Q(n6,1)
              + 2.*Q(n1,1)*Q(n3,1)*Q(n2+n4+n5,3)*Q(n6,1)-2.*Q(n1+n3,2)*Q(n2+n4+n5,3)*Q(n6,1)
              - 6.*Q(n3,1)*Q(n1+n2+n4+n5,4)*Q(n6,1)+2.*Q(n1,1)*Q(n2,1)*Q(n3+n4+n5,3)*Q(n6,1)
              - 2.*Q(n1+n2,2)*Q(n3+n4+n5,3)*Q(n6,1)-6.*Q(n2,1)*Q(n1+n3+n4+n5,4)*Q(n6,1)
              - 6.*Q(n1,1)*Q(n2+n3+n4+n5,4)*Q(n6,1)+24.*Q(n1+n2+n3+n4+n5,5)*Q(n6,1)
              - Q(n2,1)*Q(n3,1)*Q(n4,1)*Q(n5,1)*Q(n1+n6,2)+Q(n2+n3,2)*Q(n4,1)*Q(n5,1)*Q(n1+n6,2)
              + Q(n3,1)*Q(n2+n4,2)*Q(n5,1)*Q(n1+n6,2)+Q(n2,1)*Q(n3+n4,2)*Q(n5,1)*Q(n1+n6,2)
              - 2.*Q(n2+n3+n4,3)*Q(n5,1)*Q(n1+n6,2)+Q(n3,1)*Q(n4,1)*Q(n2+n5,2)*Q(n1+n6,2)
              - Q(n3+n4,2)*Q(n2+n5,2)*Q(n1+n6,2)+Q(n2,1)*Q(n4,1)*Q(n3+n5,2)*Q(n1+n6,2)
              - Q(n2+n4,2)*Q(n3+n5,2)*Q(n1+n6,2)-2.*Q(n4,1)*Q(n2+n3+n5,3)*Q(n1+n6,2)
              + Q(n2,1)*Q(n3,1)*Q(n4+n5,2)*Q(n1+n6,2)-Q(n2+n3,2)*Q(n4+n5,2)*Q(n1+n6,2)
              - 2.*Q(n3,1)*Q(n2+n4+n5,3)*Q(n1+n6,2)-2.*Q(n2,1)*Q(n3+n4+n5,3)*Q(n1+n6,2)
              + 6.*Q(n2+n3+n4+n5,4)*Q(n1+n6,2)-Q(n1,1)*Q(n3,1)*Q(n4,1)*Q(n5,1)*Q(n2+n6,2)
              + Q(n1+n3,2)*Q(n4,1)*Q(n5,1)*Q(n2+n6,2)+Q(n3,1)*Q(n1+n4,2)*Q(n5,1)*Q(n2+n6,2)
              + Q(n1,1)*Q(n3+n4,2)*Q(n5,1)*Q(n2+n6,2)-2.*Q(n1+n3+n4,3)*Q(n5,1)*Q(n2+n6,2)
              + Q(n3,1)*Q(n4,1)*Q(n1+n5,2)*Q(n2+n6,2)-Q(n3+n4,2)*Q(n1+n5,2)*Q(n2+n6,2)
              + Q(n1,1)*Q(n4,1)*Q(n3+n5,2)*Q(n2+n6,2)-Q(n1+n4,2)*Q(n3+n5,2)*Q(n2+n6,2)
              - 2.*Q(n4,1)*Q(n1+n3+n5,3)*Q(n2+n6,2)+Q(n1,1)*Q(n3,1)*Q(n4+n5,2)*Q(n2+n6,2)
              - Q(n1+n3,2)*Q(n4+n5,2)*Q(n2+n6,2)-2.*Q(n3,1)*Q(n1+n4+n5,3)*Q(n2+n6,2)
              - 2.*Q(n1,1)*Q(n3+n4+n5,3)*Q(n2+n6,2)+6.*Q(n1+n3+n4+n5,4)*Q(n2+n6,2)
              + 2.*Q(n3,1)*Q(n4,1)*Q(n5,1)*Q(n1+n2+n6,3)-2.*Q(n3+n4,2)*Q(n5,1)*Q(n1+n2+n6,3)
              - 2.*Q(n4,1)*Q(n3+n5,2)*Q(n1+n2+n6,3)-2.*Q(n3,1)*Q(n4+n5,2)*Q(n1+n2+n6,3)
              + 4.*Q(n3+n4+n5,3)*Q(n1+n2+n6,3)-Q(n1,1)*Q(n2,1)*Q(n4,1)*Q(n5,1)*Q(n3+n6,2)
              + Q(n1+n2,2)*Q(n4,1)*Q(n5,1)*Q(n3+n6,2)+Q(n2,1)*Q(n1+n4,2)*Q(n5,1)*Q(n3+n6,2)
              + Q(n1,1)*Q(n2+n4,2)*Q(n5,1)*Q(n3+n6,2)-2.*Q(n1+n2+n4,3)*Q(n5,1)*Q(n3+n6,2)
              + Q(n2,1)*Q(n4,1)*Q(n1+n5,2)*Q(n3+n6,2)-Q(n2+n4,2)*Q(n1+n5,2)*Q(n3+n6,2)
              + Q(n1,1)*Q(n4,1)*Q(n2+n5,2)*Q(n3+n6,2)-Q(n1+n4,2)*Q(n2+n5,2)*Q(n3+n6,2)
              - 2.*Q(n4,1)*Q(n1+n2+n5,3)*Q(n3+n6,2)+Q(n1,1)*Q(n2,1)*Q(n4+n5,2)*Q(n3+n6,2)
              - Q(n1+n2,2)*Q(n4+n5,2)*Q(n3+n6,2)-2.*Q(n2,1)*Q(n1+n4+n5,3)*Q(n3+n6,2)
              - 2.*Q(n1,1)*Q(n2+n4+n5,3)*Q(n3+n6,2)+6.*Q(n1+n2+n4+n5,4)*Q(n3+n6,2)
              + 2.*Q(n2,1)*Q(n4,1)*Q(n5,1)*Q(n1+n3+n6,3)-2.*Q(n2+n4,2)*Q(n5,1)*Q(n1+n3+n6,3)
              - 2.*Q(n4,1)*Q(n2+n5,2)*Q(n1+n3+n6,3)-2.*Q(n2,1)*Q(n4+n5,2)*Q(n1+n3+n6,3)
              + 4.*Q(n2+n4+n5,3)*Q(n1+n3+n6,3)+2.*Q(n1,1)*Q(n4,1)*Q(n5,1)*Q(n2+n3+n6,3)
              - 2.*Q(n1+n4,2)*Q(n5,1)*Q(n2+n3+n6,3)-2.*Q(n4,1)*Q(n1+n5,2)*Q(n2+n3+n6,3)
              - 2.*Q(n1,1)*Q(n4+n5,2)*Q(n2+n3+n6,3)+4.*Q(n1+n4+n5,3)*Q(n2+n3+n6,3)
              - 6.*Q(n4,1)*Q(n5,1)*Q(n1+n2+n3+n6,4)+6.*Q(n4+n5,2)*Q(n1+n2+n3+n6,4)
              - Q(n1,1)*Q(n2,1)*Q(n3,1)*Q(n5,1)*Q(n4+n6,2)+Q(n1+n2,2)*Q(n3,1)*Q(n5,1)*Q(n4+n6,2)
              + Q(n2,1)*Q(n1+n3,2)*Q(n5,1)*Q(n4+n6,2)+Q(n1,1)*Q(n2+n3,2)*Q(n5,1)*Q(n4+n6,2)
              - 2.*Q(n1+n2+n3,3)*Q(n5,1)*Q(n4+n6,2)+Q(n2,1)*Q(n3,1)*Q(n1+n5,2)*Q(n4+n6,2)
              - Q(n2+n3,2)*Q(n1+n5,2)*Q(n4+n6,2)+Q(n1,1)*Q(n3,1)*Q(n2+n5,2)*Q(n4+n6,2)
              - Q(n1+n3,2)*Q(n2+n5,2)*Q(n4+n6,2)-2.*Q(n3,1)*Q(n1+n2+n5,3)*Q(n4+n6,2)
              + Q(n1,1)*Q(n2,1)*Q(n3+n5,2)*Q(n4+n6,2)-Q(n1+n2,2)*Q(n3+n5,2)*Q(n4+n6,2)
              - 2.*Q(n2,1)*Q(n1+n3+n5,3)*Q(n4+n6,2)-2.*Q(n1,1)*Q(n2+n3+n5,3)*Q(n4+n6,2)
              + 6.*Q(n1+n2+n3+n5,4)*Q(n4+n6,2)+2.*Q(n2,1)*Q(n3,1)*Q(n5,1)*Q(n1+n4+n6,3)
              - 2.*Q(n2+n3,2)*Q(n5,1)*Q(n1+n4+n6,3)-2.*Q(n3,1)*Q(n2+n5,2)*Q(n1+n4+n6,3)
              - 2.*Q(n2,1)*Q(n3+n5,2)*Q(n1+n4+n6,3)+4.*Q(n2+n3+n5,3)*Q(n1+n4+n6,3)
              + 2.*Q(n1,1)*Q(n3,1)*Q(n5,1)*Q(n2+n4+n6,3)-2.*Q(n1+n3,2)*Q(n5,1)*Q(n2+n4+n6,3)
              - 2.*Q(n3,1)*Q(n1+n5,2)*Q(n2+n4+n6,3)-2.*Q(n1,1)*Q(n3+n5,2)*Q(n2+n4+n6,3)
              + 4.*Q(n1+n3+n5,3)*Q(n2+n4+n6,3)-6.*Q(n3,1)*Q(n5,1)*Q(n1+n2+n4+n6,4)
              + 6.*Q(n3+n5,2)*Q(n1+n2+n4+n6,4)+2.*Q(n1,1)*Q(n2,1)*Q(n5,1)*Q(n3+n4+n6,3)
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              - 2.*Q(n1,1)*Q(n2+n5,2)*Q(n3+n4+n6,3)+4.*Q(n1+n2+n5,3)*Q(n3+n4+n6,3)
              - 6.*Q(n2,1)*Q(n5,1)*Q(n1+n3+n4+n6,4)+6.*Q(n2+n5,2)*Q(n1+n3+n4+n6,4)
              - 6.*Q(n1,1)*Q(n5,1)*Q(n2+n3+n4+n6,4)+6.*Q(n1+n5,2)*Q(n2+n3+n4+n6,4)
              + 24.*Q(n5,1)*Q(n1+n2+n3+n4+n6,5)-Q(n1,1)*Q(n2,1)*Q(n3,1)*Q(n4,1)*Q(n5+n6,2)
              + Q(n1+n2,2)*Q(n3,1)*Q(n4,1)*Q(n5+n6,2)+Q(n2,1)*Q(n1+n3,2)*Q(n4,1)*Q(n5+n6,2)
              + Q(n1,1)*Q(n2+n3,2)*Q(n4,1)*Q(n5+n6,2)-2.*Q(n1+n2+n3,3)*Q(n4,1)*Q(n5+n6,2)
              + Q(n2,1)*Q(n3,1)*Q(n1+n4,2)*Q(n5+n6,2)-Q(n2+n3,2)*Q(n1+n4,2)*Q(n5+n6,2)
              + Q(n1,1)*Q(n3,1)*Q(n2+n4,2)*Q(n5+n6,2)-Q(n1+n3,2)*Q(n2+n4,2)*Q(n5+n6,2)
              - 2.*Q(n3,1)*Q(n1+n2+n4,3)*Q(n5+n6,2)+Q(n1,1)*Q(n2,1)*Q(n3+n4,2)*Q(n5+n6,2)
              - Q(n1+n2,2)*Q(n3+n4,2)*Q(n5+n6,2)-2.*Q(n2,1)*Q(n1+n3+n4,3)*Q(n5+n6,2)
              - 2.*Q(n1,1)*Q(n2+n3+n4,3)*Q(n5+n6,2)+6.*Q(n1+n2+n3+n4,4)*Q(n5+n6,2)
              + 2.*Q(n2,1)*Q(n3,1)*Q(n4,1)*Q(n1+n5+n6,3)-2.*Q(n2+n3,2)*Q(n4,1)*Q(n1+n5+n6,3)
              - 2.*Q(n3,1)*Q(n2+n4,2)*Q(n1+n5+n6,3)-2.*Q(n2,1)*Q(n3+n4,2)*Q(n1+n5+n6,3)
              + 4.*Q(n2+n3+n4,3)*Q(n1+n5+n6,3)+2.*Q(n1,1)*Q(n3,1)*Q(n4,1)*Q(n2+n5+n6,3)
              - 2.*Q(n1+n3,2)*Q(n4,1)*Q(n2+n5+n6,3)-2.*Q(n3,1)*Q(n1+n4,2)*Q(n2+n5+n6,3)
              - 2.*Q(n1,1)*Q(n3+n4,2)*Q(n2+n5+n6,3)+4.*Q(n1+n3+n4,3)*Q(n2+n5+n6,3)
              - 6.*Q(n3,1)*Q(n4,1)*Q(n1+n2+n5+n6,4)+6.*Q(n3+n4,2)*Q(n1+n2+n5+n6,4)
              + 2.*Q(n1,1)*Q(n2,1)*Q(n4,1)*Q(n3+n5+n6,3)-2.*Q(n1+n2,2)*Q(n4,1)*Q(n3+n5+n6,3)
              - 2.*Q(n2,1)*Q(n1+n4,2)*Q(n3+n5+n6,3)-2.*Q(n1,1)*Q(n2+n4,2)*Q(n3+n5+n6,3)
              + 4.*Q(n1+n2+n4,3)*Q(n3+n5+n6,3)-6.*Q(n2,1)*Q(n4,1)*Q(n1+n3+n5+n6,4)
              + 6.*Q(n2+n4,2)*Q(n1+n3+n5+n6,4)-6.*Q(n1,1)*Q(n4,1)*Q(n2+n3+n5+n6,4)
              + 6.*Q(n1+n4,2)*Q(n2+n3+n5+n6,4)+24.*Q(n4,1)*Q(n1+n2+n3+n5+n6,5)
              + 2.*Q(n1,1)*Q(n2,1)*Q(n3,1)*Q(n4+n5+n6,3)-2.*Q(n1+n2,2)*Q(n3,1)*Q(n4+n5+n6,3)
              - 2.*Q(n2,1)*Q(n1+n3,2)*Q(n4+n5+n6,3)-2.*Q(n1,1)*Q(n2+n3,2)*Q(n4+n5+n6,3)
              + 4.*Q(n1+n2+n3,3)*Q(n4+n5+n6,3)-6.*Q(n2,1)*Q(n3,1)*Q(n1+n4+n5+n6,4)
              + 6.*Q(n2+n3,2)*Q(n1+n4+n5+n6,4)-6.*Q(n1,1)*Q(n3,1)*Q(n2+n4+n5+n6,4)
              + 6.*Q(n1+n3,2)*Q(n2+n4+n5+n6,4)+24.*Q(n3,1)*Q(n1+n2+n4+n5+n6,5)
              - 6.*Q(n1,1)*Q(n2,1)*Q(n3+n4+n5+n6,4)+6.*Q(n1+n2,2)*Q(n3+n4+n5+n6,4)
              + 24.*Q(n2,1)*Q(n1+n3+n4+n5+n6,5)+24.*Q(n1,1)*Q(n2+n3+n4+n5+n6,5)
              - 120.*Q(n1+n2+n3+n4+n5+n6,6);
   TComplex *out = (TComplex*) &formula;
   return out;
```
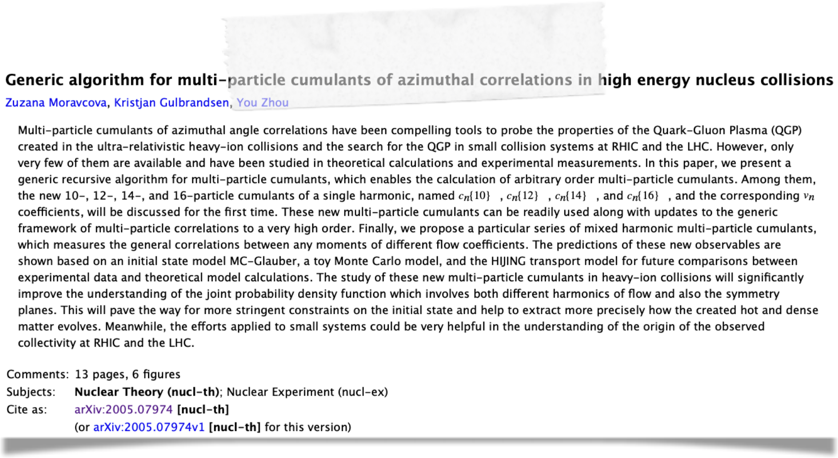

Generic algorithm

- ❖ 2020, Generic algorithm method, the most *efficient*, *precise* and *reliable* method

arXiv: 2005.07974

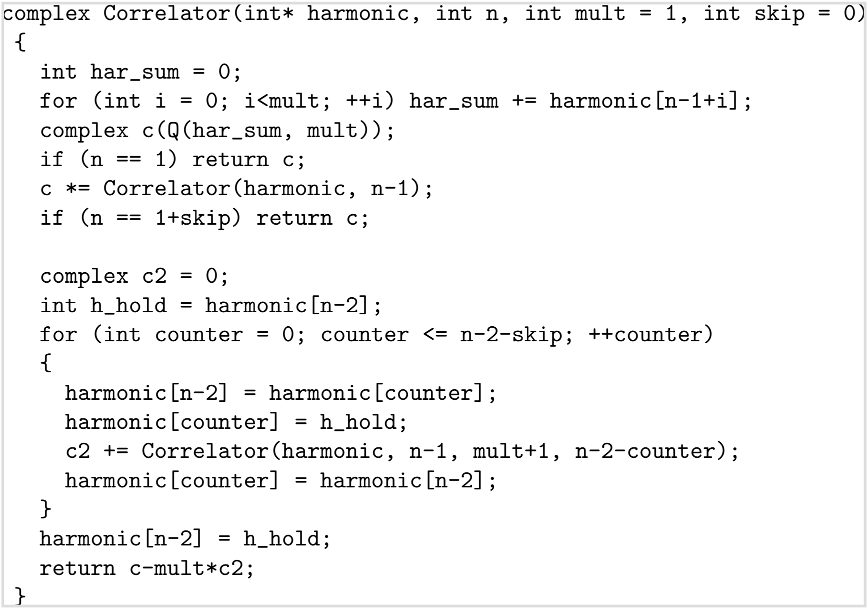


Generic algorithm for multi-particle cumulants of azimuthal correlations in high energy nucleus collisions

Zuzana Moravcova, Kristjan Gulbrandsen, You Zhou

Multi-particle cumulants of azimuthal angle correlations have been compelling tools to probe the properties of the Quark-Gluon Plasma (QGP) created in the ultra-relativistic heavy-ion collisions and the search for the QGP in small collision systems at RHIC and the LHC. However, only very few of them are available and have been studied in theoretical calculations and experimental measurements. In this paper, we present a generic recursive algorithm for multi-particle cumulants, which enables the calculation of arbitrary order multi-particle cumulants. Among them, the new 10-, 12-, 14-, and 16-particle cumulants of a single harmonic, named $c_n\{10\}$, $c_n\{12\}$, $c_n\{14\}$, and $c_n\{16\}$, and the corresponding v_n coefficients, will be discussed for the first time. These new multi-particle cumulants can be readily used along with updates to the generic framework of multi-particle correlations to a very high order. Finally, we propose a particular series of mixed harmonic multi-particle cumulants, which measures the general correlations between any moments of different flow coefficients. The predictions of these new observables are shown based on an initial state model MC-Glauber, a toy Monte Carlo model, and the HIJING transport model for future comparisons between experimental data and theoretical model calculations. The study of these new multi-particle cumulants in heavy-ion collisions will significantly improve the understanding of the joint probability density function which involves both different harmonics of flow and also the symmetry planes. This will pave the way for more stringent constraints on the initial state and help to extract more precisely how the created hot and dense matter evolves. Meanwhile, the efforts applied to small systems could be very helpful in the understanding of the origin of the observed collectivity at RHIC and the LHC.

Comments: 13 pages, 6 figures
Subjects: **Nuclear Theory (nucl-th)**; Nuclear Experiment (nucl-ex)
Cite as: arXiv:2005.07974 **[nucl-th]**
(or arXiv:2005.07974v1 **[nucl-th]** for this version)



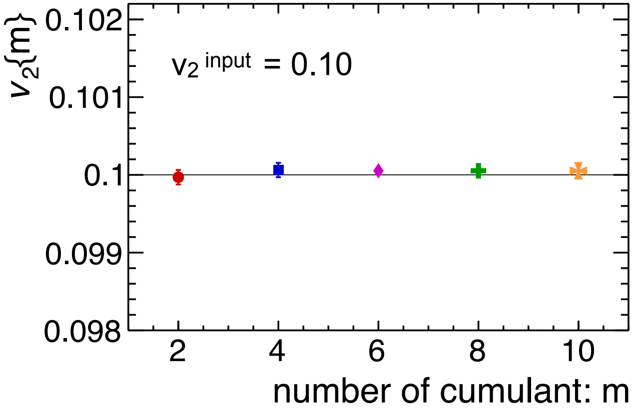
```
complex Correlator(int* harmonic, int n, int mult = 1, int skip = 0)
 {
   int har_sum = 0;
   for (int i = 0; i<mult; ++i) har_sum += harmonic[n-1+i];
   complex c(Q(har_sum, mult));
   if (n == 1) return c;
   c *= Correlator(harmonic, n-1);
   if (n == 1+skip) return c;

   complex c2 = 0;
   int h_hold = harmonic[n-2];
   for (int counter = 0; counter <= n-2-skip; ++counter)
   {
     harmonic[n-2] = harmonic[counter];
     harmonic[counter] = h_hold;
     c2 += Correlator(harmonic, n-1, mult+1, n-2-counter);
     harmonic[counter] = harmonic[n-2];
   }
   harmonic[n-2] = h_hold;
   return c-mult*c2;
 }
```

- Few lines of code, for **any** multi-particle correlations
- Much faster than generic framework (much shorter CPU times)

Feel free to contact you.zhou@cern.ch
if you have any technical question

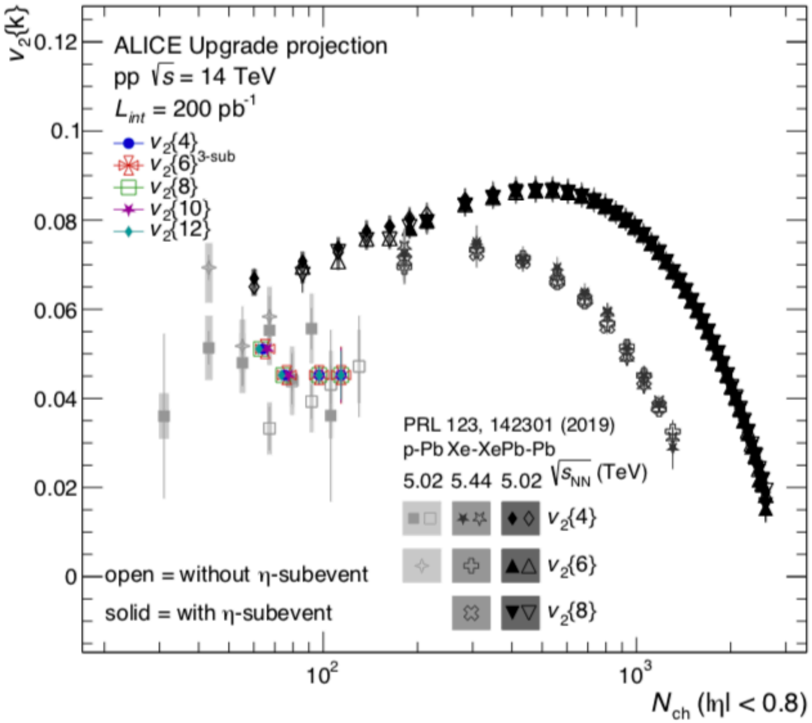
New possibilities

arXiv: 2005.07974



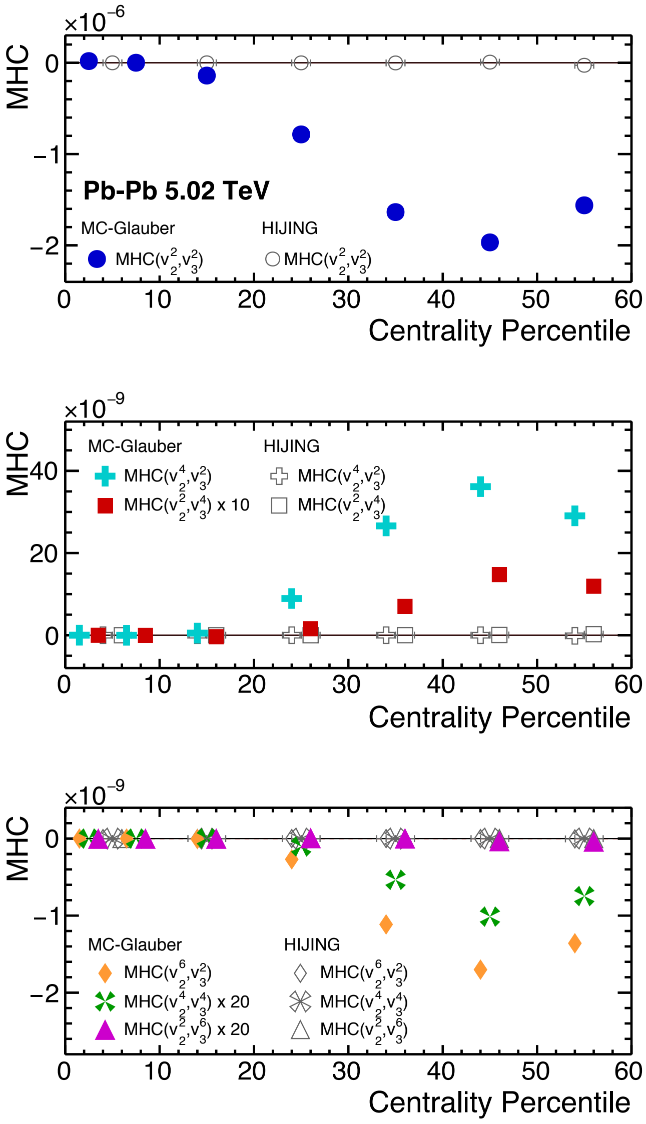
- Validation successful for multi-particle cumulants

`“Future high-energy pp programme with ALICE”,` `ALICE-PUBLIC-2020-005`



New possibility of m-particle correlations (m>8) with existing RHIC & LHC heavy-ion data, and also in the future HL-LHC program.

With many more observables



arXiv: 2005.07974

- New *generic algorithm* of multi-particle mixed harmonic cumulants, which for the first time study the correlations between v_2^k, v_3^l, v_4^p
  - Free of non-flow (Results from HIJING are all zero)
  - Sensitive to the initial conditions and transport properties of QGP (based on hydro calculations)
  - negative, positive and negative signs of correlations between 4-, 6- and 8-particle mixed harmonic cumulants ***MHC***.

Summary

- ❖ “***generic algorithm***”, one code for all flow studies

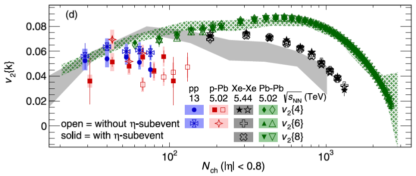
EXP data

LHC, RHIC,
SPS, FAIR
NICA

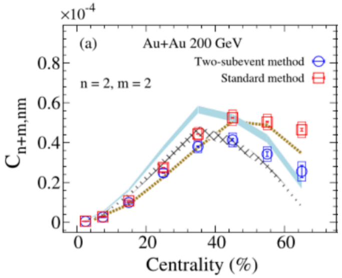
TH simulations

Generic algorithm

ALICE, PRL123, 142301 (2019)



STAR, PLB809 (2020) 135728



Thanks for your attentions!

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Ultra-higher order cumulants

arXiv: 2005.07974

- 14-particle cumulant of single harmonic

$$\begin{aligned} v_n\{14\}^{14} = \frac{1}{274800}\Big(& \langle v_n^{14}\rangle - 49\,\langle v_n^{12}\rangle\,\langle v_n^{2}\rangle \\ & -441\,\langle v_n^{10}\rangle\,\langle v_n^{4}\rangle + 1764\,\langle v_n^{10}\rangle\,\langle v_n^{2}\rangle^2 \\ & -1225\,\langle v_n^{8}\rangle\,\langle v_n^{6}\rangle + 22050\,\langle v_n^{8}\rangle\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle \\ & -44100\,\langle v_n^{8}\rangle\,\langle v_n^{2}\rangle^3 + 19600\,\langle v_n^{6}\rangle^2\,\langle v_n^{2}\rangle \\ & +44100\,\langle v_n^{6}\rangle\,\langle v_n^{4}\rangle^2 - 529200\,\langle v_n^{6}\rangle\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle^2 \\ & +705600\,\langle v_n^{6}\rangle\,\langle v_n^{2}\rangle^4 - 396900\,\langle v_n^{4}\rangle^3\,\langle v_n^{2}\rangle \\ & +3175200\,\langle v_n^{4}\rangle^2\,\langle v_n^{2}\rangle^3 - 6350400\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle^5 \\ & +3628800\,\langle v_n^{2}\rangle^7 \Big) \end{aligned}$$

- 16-particle cumulant of single harmonic

$$\begin{aligned} v_n\{16\}^{16} = -\frac{1}{10643745}\Big(& \langle v_n^{16}\rangle - 64\,\langle v_n^{14}\rangle\,\langle v_n^{2}\rangle \\ & -784\,\langle v_n^{12}\rangle\,\langle v_n^{4}\rangle + 3136\,\langle v_n^{12}\rangle\,\langle v_n^{2}\rangle^2 \\ & -3136\,\langle v_n^{10}\rangle\,\langle v_n^{6}\rangle + 56448\,\langle v_n^{10}\rangle\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle \\ & -112896\,\langle v_n^{10}\rangle\,\langle v_n^{2}\rangle^3 - 2450\,\langle v_n^{8}\rangle^2 \\ & +156800\,\langle v_n^{8}\rangle\,\langle v_n^{6}\rangle\,\langle v_n^{2}\rangle + 176400\,\langle v_n^{8}\rangle\,\langle v_n^{4}\rangle^2 \\ & -2116800\,\langle v_n^{8}\rangle\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle^2 \\ & +2822400\,\langle v_n^{8}\rangle\,\langle v_n^{2}\rangle^4 + 313600\,\langle v_n^{6}\rangle^2\,\langle v_n^{4}\rangle \\ & -1881600\,\langle v_n^{6}\rangle^2\,\langle v_n^{2}\rangle^2 \\ & -8467200\,\langle v_n^{6}\rangle\,\langle v_n^{4}\rangle^2\,\langle v_n^{2}\rangle \\ & +45158400\,\langle v_n^{6}\rangle\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle^3 \\ & -45158400\,\langle v_n^{6}\rangle\,\langle v_n^{2}\rangle^5 - 1587600\,\langle v_n^{4}\rangle^4 \\ & +50803200\,\langle v_n^{4}\rangle^3\,\langle v_n^{2}\rangle^2 \\ & -254016000\,\langle v_n^{4}\rangle^2\,\langle v_n^{2}\rangle^4 \\ & +406425600\,\langle v_n^{4}\rangle\,\langle v_n^{2}\rangle^6 - 203212800\,\langle v_n^{2}\rangle^8 \Big) \end{aligned}$$