The high-frequency gravitational waves in exact inflationary models with Gauss-Bonnet term

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Standard inflation

The Einstein gravity and scalar field (inflaton)

$$S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{1}$$

Varying this action with respect to $g_{\mu\nu}$ and ϕ in a spatially flat Friedmann-Robertson-Walker universe with a scale factor a

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{2}$$

yields the background equations for the standard inflation (**J. D. Barrow**)

$$3\overline{H}^2 = \frac{1}{2}(\dot{\phi})^2 + V(\phi) \tag{3}$$

$$\frac{1}{2}(\dot{\phi})^2 = -\dot{\overline{H}} \tag{4}$$

Slow-roll approximation

The condition $V(\phi) \gg \frac{1}{2}(\dot{\phi})^2$

$$3\overline{H}^2 \approx V(\phi) \tag{5}$$

$$\frac{1}{2}(\dot{\phi})^2 = -\dot{\overline{H}} \tag{6}$$

Restrict the form of potential (flat potential) (**A.R. Liddle and D.H. Lyth**)

Restrict the GWB spectrum (low-frequencies) (**M. Maggiore; E. Komatsu**)

Gauss-Bonnet Term

$$R^2_{\mathrm{GB}} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

The simplest correction term in the low-energy effective action of the heterotic string (**Zwiebach B., Zumino B.**)

The second order of Lovelock gravity (**Lovelock D.**), $D > 4$ in the case of minimal coupling

Nonsingular cosmological solutions with Gauss-Bonnet Term (**S. Tsujikawa, R. Brandenberger and F. Finelli**)

Inflation with GB correction

Now, we consider an action with the Gauss-Bonnet term that is coupled to a scalar field

$$S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{2}\xi(\phi)R^2_{\rm GB}\right], \tag{7}$$

The Gauss-Bonnet coupling $\xi(\phi)$ is required to be a function of a scalar field in order to give nontrivial effects on the background dynamics.
Varying this action with respect to $g_{\mu\nu}$ and ϕ yields the Einstein and field equation in a spatially flat Friedmann-Robertson-Walker universe

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + 2V) + T^{GB}_{\mu\nu}\right), \tag{8}$$

$$\Box\phi - V_\phi - \frac{1}{2}T^{GB} = 0, \tag{9}$$

$$
\begin{aligned}
T^{GB}_{\mu\nu} =& 4(\partial^\rho \partial^\sigma \xi R_{\mu\rho\nu\sigma} - \Box \xi R_{\mu\nu} + 2\partial_\rho \partial_{(\mu} \xi R^\rho{}_{\nu)} \\
-& \frac{1}{2}\partial_\mu \partial_\nu \xi R) - 2(2\partial_\rho \partial_\sigma \xi R^{\rho\sigma} - \Box \xi R) g_{\mu\nu}, \qquad (10) \\
T^{GB} =& \xi_\phi R^2_{GB}. \qquad (11)
\end{aligned}
$$

The background Einstein and field equations yield

$$
H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + 12\dot{\xi}H^3\right), \qquad (12)
$$

$$
\dot{H} = -\frac{1}{2}\left(\dot{\phi}^2 - 4\ddot{\xi}H^2 - 4\dot{\xi}H(2\dot{H} - H^2)\right), \qquad (13)
$$

$$
\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + 12\xi_{,\phi}H^2\left(\dot{H} + H^2\right) = 0, \qquad (14)
$$

The equation (14) is the result of the equations (12)–(13). When $\xi = const$ we have the standard inflation.

The method of exact solutions

The exact solutions for $D > 4$ (**D. Chirkov , A. Toporensky ; A. Makarenko**). Earlier, inflation for a single scalar field with nonminimal coupling to the Gauss-Bonnet term in 4-dimensional FRW universe was considered on the basis of slow-roll approximation, only (**Z. Guo, D. J. Schwarz; S. Koh et. al.**).

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + 12\dot{\xi}H^3\right), \qquad (15)$$

$$\dot{H} = -\frac{1}{2}\left(\dot{\phi}^2 - 4\ddot{\xi}H^2 - 4\dot{\xi}H(2\dot{H} - H^2)\right) \qquad (16)$$

Now, we define the connection between standard inflation and inflation with GB correction

$$\overline{H} = H(1 - 2\dot{\xi}H) \qquad (17)$$

When $\xi = const$, the equations (12)–(13) are reduced to (3)–(4) and $H = \overline{H}$.

The equations (12)–(13), in this case, are written as

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = -3H^2 + 6\overline{H}H \tag{18}$$

$$\frac{1}{2}\dot{\phi}^2 = -\dot{\overline{H}} - \overline{H}H + H^2 \tag{19}$$

In the case of $\xi = const$, $H = \overline{H}$ and equations (18)–(19) are reduced to (3)–(4). Thus, $\overline{H}$ is the Hubble parameter for standard inflationary models.
Now, we write the equations (18)–(19) in the following form

$$V = -4H^2 + 7\overline{H}H + \dot{\overline{H}} \tag{20}$$

$$\frac{1}{2}\dot{\phi}^2 = -\dot{\overline{H}} - \overline{H}H + H^2 \tag{21}$$

Further, by selecting the Hubble parameter $H = H(t)$ and the scalar field $\phi = \phi(t)$ we will generate the exact solutions of equations (20)–(21).

De Sitter expansion

The scale factor

$$a(t) = a_0 \exp(At) \tag{22}$$

Consider the inflationary model with

$$H(t) = A \tag{23}$$
$$\phi(t) = B \exp(-At), \tag{24}$$

where A and B are the positive constants .

The exact solutions of the equations (20)–(21) and (17) are:

$$\overline{H}(t) = AB^2 \exp(-2At) + A \tag{25}$$
$$V(\phi) = 5A^2\phi^2 + 3A^2 \tag{26}$$
$$\xi(\phi) = \frac{\phi^2}{4A^2} \tag{27}$$

Power-law inflation

We consider the inflationary model with

$$H(t) = B/t \tag{28}$$

$$\phi(t) = \pm\sqrt{B^2 + A - AB}\ \ln(t) + \phi_0, \tag{29}$$

where A, B are the positive constants.
The exact solutions of the equations (20)–(21) and (17) are:

$$\overline{H}(t) = A/t \tag{30}$$

$$V(\phi) = (-4B^2 + 7AB - A)\exp(\pm 2(\phi - \phi_0)/C) \tag{31}$$

$$\xi(\phi) = \frac{B - A}{4B^2}\exp(\mp 2(\phi - \phi_0)/C), \tag{32}$$

where $C = \sqrt{B^2 + A - AB}$.
Thus, we obtain $\overline{H} = \alpha H$, where $\alpha = A/B$.

Double-well potential

Consider the inflationary model with

$$H(t) = C\exp(-At), \tag{33}$$

$$\phi(t) = B\exp\left(-\frac{A}{2}t\right) \tag{34}$$

The exact solutions of the equations (20)–(21) and (17) are:

$$\overline{H}(t) = -\frac{1}{4}\frac{A^2B^2}{C} + A + C\exp(-At) \tag{35}$$

$$V(\phi) = \frac{1}{4B^2}\phi^2\left(12\frac{C^2}{B^2}\phi^2 + 24AC - 7A^2B^2\right) \tag{36}$$

$$\xi(\phi) = \frac{(AB^2 - 4C)B^4}{16C^3}\phi^{-4} \tag{37}$$

Thus, we have $\overline{H} = H + \beta$, where $\beta = -\frac{1}{4}\frac{A^2B^2}{C} + A$.

Cosmological perturbations

During inflation, quantum fluctuations of the scalar field will create a metric perturbation. In a linear order, for the case of standard inflation for Fourier modes scalar v_k and tensor u_k perturbation, Mukhanov-Sasaki equations are written as follows (**V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger**)

$$\frac{d^2 v_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\eta^2}\right) v_k = 0 \tag{38}$$

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{a}\frac{d^2 a}{d\eta^2}\right) u_k = 0, \tag{39}$$

where $z = a\dot{\phi}/H$, k – wavenumber, η – conformal time.

The exact cosmological parameters at the crossing of the Hubble radius $(k = aH)$ (**S. V. Chervon, I. V. Fomin**)

$$r = -4\frac{\dot{\overline{H}}}{\overline{H}^2} \tag{40}$$

$$\mathcal{P}_{\mathcal{R}}(k) = -\frac{\overline{H}^4}{8\pi^2\dot{\overline{H}}} \tag{41}$$

$$\mathcal{P}_{\mathcal{G}}(k) = \frac{\overline{H}^2}{2\pi^2} \tag{42}$$

$$n_S(k) - 1 = \frac{4\dot{\overline{H}} - \frac{\overline{H}\,\ddot{\overline{H}}}{\dot{\overline{H}}}}{\dot{\overline{H}} + \overline{H}^2} \tag{43}$$

$$n_{\mathcal{G}}(k) = \frac{2\dot{\overline{H}}}{\dot{\overline{H}} + \overline{H}^2} \tag{44}$$

On the basis of the connection $\overline{H} = H(1 - 2\dot{\xi}H)$ one can calculate the exact values of parameters of cosmological perturbations for inflation with GB correction from this formulas.
For power-law inflation $\overline{H} = \alpha H$ we have:

$$r = \frac{4}{\alpha B} \tag{45}$$

$$\mathcal{P}_S = \frac{\alpha^3 B^3}{8\pi^2 t_*^2} \tag{46}$$

$$\mathcal{P}_T = \frac{\alpha^2 B^2}{2\pi^2 t_*^2} \tag{47}$$

$$n_S - 1 = \frac{2}{1 - \alpha B} \tag{48}$$

$$n_T = \frac{2}{1 - \alpha B}, \tag{49}$$

where t_* is the time of Hubble radius crossing.

For double-well potential $\overline{H} = H + \beta$ we have

$$r = \frac{-4\dot{H}}{(H+\beta)^2} \tag{50}$$

$$\mathcal{P}_{\mathcal{R}}(k) = -\frac{(H+\beta)^4}{8\pi^2\dot{H}} \tag{51}$$

$$\mathcal{P}_{\mathcal{G}}(k) = \frac{(H+\beta)^2}{2\pi^2} \tag{52}$$

$$n_S(k) - 1 = \frac{4\dot{H} - \frac{(H+\beta)\,\ddot{H}}{\dot{H}}}{\dot{H} + (H+\beta)^2} \tag{53}$$

$$n_{\mathcal{G}}(k) = \frac{2\dot{H}}{\dot{H} + (H+\beta)^2} \tag{54}$$

The high-frequency gravitational waves

The amplitude of gravitational waves at the end of the inflation in the selected unit system is defined as follows (**V. Sahni, M. Sami and T. Souradeep**)

$$h_{GW}^2 = \frac{\overline{H}(t = t_i)}{4\pi^2}$$

The spectral energy density of gravitational waves

$$\Omega_{GW}^{MD}(f) = \frac{3}{8\pi^2} h_{GW}^2 \Omega_{mq} \left(\frac{f_q}{f}\right), f_q \leq f < f_{MD} \quad (55)$$

$$\Omega_{GW}^{RD}(f) = \frac{1}{6\pi} h_{GW}^2 \Omega_{rq}, f_{MD} \leq f < f_{RD} \quad (56)$$

$$\Omega_{GW}^{kin}(f) = \frac{3}{8\pi^2} h_{GW}^2 \Omega_{mq} \left(\frac{f}{f_{RD}}\right), f_{RD} \leq f < f_{kin}, \quad (57)$$

where f_q, f_{kin}, f_{RD} f_{MD} – the frequencies of the gravitational waves at each stage of evolution of the Universe

$$f_q = \frac{1}{2} H_q \tag{58}$$

$$f_{MD} = \frac{3}{2\pi} f_q \left(\frac{\Omega_{mq}}{\Omega_{rq}} \right)^{1/2} \tag{59}$$

$$f_{RD} = \frac{1}{4} f_q \left(\frac{\Omega_{rq}}{\Omega_{mq}} \right)^{1/2} \frac{T_{rh}}{T_{MD}} \tag{60}$$

$$f_{kin} = H_{kin} \left(\frac{T_q}{T_{rh}} \right) \left(\frac{H_{rh}}{H_{kin}} \right)^{1/3} \tag{61}$$

Here $H_q = 67.8 \pm 0.9 kms^{-1} Mpc^{-1}$, $\Omega_{mq} = 0.308 \pm 0.012$ and $\Omega_{rq} = (9.230 \pm 0.022) \times 10^{-5}$ are Hubble parameter, the density of matter and radiation at the modern era, $T_{rh} = 1 \times 10^{14} GeV$ and H_{rh} – reheating temperature and Hubble parameter, that we take approximately same as the temperature and Hubble parameter at the end of inflation.

The possibility of experimental detection of relic gravitational waves

To date, several projects of searching for gravitational waves, such as projects LIGO (USA), VIRGO (Italy, France), TAMA-300 (Japan), GEO-600 (Germany) and others, are realized.

One of the promising methods for increasing the sensitivity of gravitational antennas in the high frequency part of the spectrum is the use of low-frequency optical resonance phenomenon (LOR) in the Fabry-Perot interferometer (**A. Morozov**)

Minimum detectable spectral density fluctuations of the space-time by using low-frequency optical resonance in a Fabry-Perot interferometer can be estimated by the formula

$$G_h(f) > \sqrt{\frac{2\pi\kappa}{c^2 T\Delta}} \frac{2\pi\hbar f^{3/2}}{k_e W_0} \qquad (62)$$

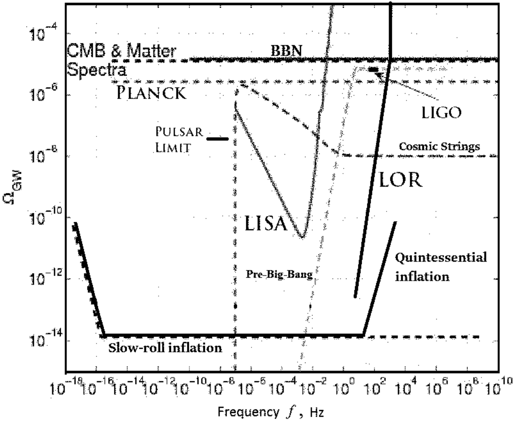
where κ – phase shift that characterizes the setting of the interferometer , c – the speed of light , T – time averaging of the spectral density , Δ – loss per cycle reflections , k_e – wave number , W_0 – power incident on the Fabry-Perot interferometer monochromatic laser radiation , f - gravitational wave frequency.

To move to the value of the energy density of gravitational waves $\Omega_{GW}(f)$, one can use the formula

$$\Omega_{GW}(f) = \frac{4\pi^2}{3H_q^2} f^3 G_h(f) \tag{63}$$

where H_q – the Hubble parameter. Substituting formula (62) in the expression (63) gives an estimate for the minimum energy density of gravitational waves that can be detected with the use of low-frequency optical resonance

$$\Omega_{GW}(f) > \sqrt{\frac{2\pi\kappa}{c^2 T \Delta}} \frac{8\pi^3 \hbar f^{9/2}}{3H_q^2 k_e W_0} \tag{64}$$



$\Omega_{GW}(f)$ for various experiments and for the case of low-frequency optical resonance with the following setup switches: $\kappa = \Delta$, $T = 10^7 sec$ (116 a day), $k_e = 5.9 \cdot 10^6 m^{-1}$, $(\lambda_e = 1.064 \mu m)$, $W_0 = 10^4$ W.