The screening Horndeski cosmologies

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In 1974, Horndeski derived the action of the most general scalar-tensor theories with second-order equations of motion

\[ \text{Horndeski Lagrangian:} \]

\[
L_H = \sqrt{-g} \left( L_2 + L_3 + L_4 + L_5 \right)
\]

\[
L_2 = G_2(X, \Phi),
\]

\[
L_3 = G_3(X, \Phi) \Box \Phi,
\]

\[
L_4 = G_4(X, \Phi) R + \partial_X G_4(X, \Phi) \delta^{\mu\nu}_{\alpha\beta} \nabla_\mu \Phi \nabla_\nu \Phi,
\]

\[
L_5 = G_5(X, \Phi) G_{\mu\nu} \nabla^{\mu\nu} \Phi - \frac{1}{6} \partial_X G_5(X, \Phi) \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \nabla_\mu \Phi \nabla_\nu \Phi \nabla_\rho \Phi,
\]

where \( X = -\frac{1}{2} (\nabla \phi)^2 \), and \( G_k(X, \Phi) \) are arbitrary functions,

and \( \delta^{\lambda\rho}_{\nu\alpha} = 2! \delta^{\lambda}_{[\nu} \delta^{\rho}_{\alpha]} \), \( \delta^{\lambda\rho\sigma}_{\nu\alpha\beta} = 3! \delta^{\lambda}_{[\nu} \delta^{\rho}_{\alpha} \delta^{\sigma}_{\beta]} \).
There is a special subclass of the theory, sometimes called Fab Four (F4), for which the coefficients are chosen such that the Lagrangian becomes

\[ L_{F4} = \sqrt{-g} (\mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda) \]

with

\[ \mathcal{L}_J = V_J(\Phi) G_{\mu\nu} \nabla^\mu \Phi \nabla^\nu \Phi, \]
\[ \mathcal{L}_P = V_P(\Phi) P_{\mu\nu\rho\sigma} \nabla^\mu \Phi \nabla^\rho \Phi \nabla^\nu \Phi, \]
\[ \mathcal{L}_G = V_G(\Phi) R, \]
\[ \mathcal{L}_R = V_R(\Phi) (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2). \]

Here the double dual of the Riemann tensor is

\[ P_{\alpha\beta}^{\mu\nu} = -\frac{1}{4} \delta^{\mu\nu\gamma\delta} R_{\sigma\lambda}^{\gamma\delta} R_{\alpha\beta}^{\sigma\lambda} R_{\mu\nu} - 2R_{[\alpha}^{\mu} \delta^{\nu]}_{\beta]} - 2R^{\nu}_{[\alpha} \delta^{\mu}_{\beta]} - R_{\delta^{\mu}_{[\alpha} \delta^{\nu}_{\beta]}}, \]

whose contraction is the Einstein tensor, \( P^{\mu\nu}_{\alpha\nu} = G^{\mu}_{\nu}. \)
Fab Four subclass of the Horndeski theory

Fab Four Lagrangian:

\[ L_{F4} = \sqrt{-g} (L_J + L_P + L_G + L_R - 2\Lambda) \]

- The Fab Four model is distinguished by the *screening property* – it is the most general subclass of the Horndeski theory in which flat space is a solution, despite the presence of the cosmological term \( \Lambda \).
- This property suggests that \( \Lambda \) is actually irrelevant and hence there is no need to explain its value.
- Indeed, however large \( \Lambda \) is, Minkowski space is always a solution and so one may hope that a slowly accelerating universe will be a solution as well.
The model with nonminimal kinetic coupling

Action of the theory with nonminimal kinetic coupling:

\[
S = \frac{1}{2} \int \left( M_{Pl}^2 R - (\alpha G_{\mu\nu} + \varepsilon g_{\mu\nu}) \nabla^{\mu} \Phi \nabla^{\nu} \Phi - 2\Lambda \right) \sqrt{-g} \, d^4x + S_m
\]

The gravitational equations:

\[
M_{Pl}^2 G_{\mu\nu} + \Lambda g_{\mu\nu} = \alpha T_{\mu\nu} + \varepsilon T^{(\Phi)}_{\mu\nu} + T^{(m)}_{\mu\nu},
\]

with

\[
T_{\mu
u} = \mathcal{P}_{\alpha\mu\nu\beta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi + \frac{1}{2} g_{\mu\lambda} \delta^{\lambda}_{\nu\alpha\beta} \nabla^{\alpha} \Phi \nabla^{\beta} \Phi - XG_{\mu\nu},
\]

\[
T^{(\Phi)}_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi + X g_{\mu\nu},
\]

\[
T^{(m)}_{\mu\nu} = (\rho + p) U_\mu U_\nu + pg_{\mu\nu},
\]

The scalar equation

\[
\nabla_\mu ((\alpha G^{\mu\nu} + \varepsilon g^{\mu\nu}) \nabla_\nu \Phi) = 0.
\]
Cosmological models

The FLRW ansatz for the metric:

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right], \]

where \( a(t) \) is the cosmological factor, \( H = \dot{a}/a \) is the Hubble parameter.

Gravitational equations:

\[-3M_{\text{Pl}}^2 \left( H^2 + \frac{K}{a^2} \right) + \frac{1}{2} \varepsilon \psi^2 - \frac{3}{2} \alpha \psi^2 \left( 3H^2 + \frac{K}{a^2} \right) + \Lambda + \rho = 0, \]
\[-M_{\text{Pl}}^2 \left( 2\dot{H} + 3H^2 + \frac{K}{a^2} \right) - \frac{1}{2} \varepsilon \psi^2 - \alpha \psi^2 \left( \dot{H} + \frac{3}{2} H^2 - \frac{K}{a^2} + 2H\frac{\dot{\psi}}{\psi} \right) + \Lambda - p = 0. \]

The scalar field equation:

\[ \frac{1}{a^3} \frac{d}{dt} \left( a^3 \left( 3\alpha \left( H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi \right) = 0, \]

where \( \psi = \dot{\Phi} \), and \( \Phi = \Phi(t) \) is a homogeneous scalar field.
The first integral of the scalar field equation:

$$a^3 \left(3\alpha \left(H^2 + \frac{K}{a^2}\right) - \varepsilon\right) \psi = C,$$

where $C$ is the Noether charge associated with the shift symmetry $\Phi \to \Phi + \Phi_0$.

Let $C = 0$. One finds in this case two different solutions:

**GR branch:** $\psi = 0 \implies H^2 + \frac{K}{a^2} = \frac{\Lambda + \rho}{3M_{Pl}^2}$

**Screening branch:** $H^2 + \frac{K}{a^2} = \frac{\varepsilon}{3\alpha} \implies \psi^2 = \frac{\alpha (\Lambda + \rho) - \varepsilon M_{Pl}^2}{\alpha (\varepsilon - 3\alpha K/a^2)}$

**NOTICE:** The role of the cosmological constant in the screening solution is played by $\varepsilon/3\alpha$ while the $\Lambda$-term is screened and makes no contribution to the universe acceleration.

Note also that the matter density $\rho$ is screened in the same sense.
Cosmological models

Let $C \neq 0$, then

$$\psi = \frac{C}{a^3 \left[ 3\alpha \left( H^2 + \frac{K}{a^2} \right) - \varepsilon \right]} ,$$

and the modified Friedmann equation reads

$$3M_{\text{Pl}}^2 \left( H^2 + \frac{K}{a^2} \right) = \frac{C^2 \left[ \varepsilon - 3\alpha \left( 3H^2 + \frac{K}{a^2} \right) \right]}{2a^6 \left[ \varepsilon - 3\alpha \left( H^2 + \frac{K}{a^2} \right) \right]^2} + \Lambda + \rho .$$

Introducing dimensionless values and density parameters

$$H^2 = H_0^2 y, \quad a = a_0 a, \quad \rho_{\text{cr}} = 3M_{\text{Pl}}^2 H_0^2, \quad \zeta = \frac{\varepsilon}{3\alpha H_0^2} ,$$

$$\Omega_0 = \frac{\Lambda}{\rho_{\text{cr}}}, \quad \Omega_2 = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_6 = \frac{C^2}{6\alpha a_0^6 H_0^2 \rho_{\text{cr}}}, \quad \rho = \rho_{\text{cr}} \left( \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} \right)$$

gives

the master equation:

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6 \left[ \zeta - 3y + \frac{\Omega_2}{a^2} \right]}{a^6 \left[ \zeta - y + \frac{\Omega_2}{a^2} \right]^2}.$$
Asymptotical behavior: Late time limit $a \rightarrow \infty$

**GR branch:**

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{(\zeta - 3\Omega_0)\Omega_6}{(\Omega_0 - \zeta)^2 a^6} + \mathcal{O}\left(\frac{1}{a^7}\right) \Rightarrow H^2 \rightarrow \Lambda/3$$

Notice: The GR solution is stable (no ghost) if and only if $\zeta > \Omega_0$.

**Screening branches:**

$$y_{\pm} = \zeta + \frac{\Omega_2}{a^2} \pm \frac{\chi}{(\Omega_0 - \zeta) a^3} \pm \frac{\Omega_2 \Omega_6}{\chi a^5} - \frac{\Omega_6 (\zeta - 3\Omega_0) \pm \Omega_3 \chi}{2(\Omega_0 - \zeta)^2 a^6} + \mathcal{O}\left(\frac{1}{a^7}\right)$$

$$\Rightarrow \quad H^2 \rightarrow \epsilon/3\alpha$$

Notice: The screening solutions are stable (no ghost) if and only if $0 < \zeta < \Omega_0$. 

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Asymptotical behavior: The limit $a \to 0$

**GR branch:**

$$y = \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} + \frac{\Omega_2 \Omega_4 - 3\Omega_6}{\Omega_4 a^2} + \frac{3\Omega_3 \Omega_6}{\Omega_4 a} + \mathcal{O}(1)$$

**Notice:** The GR solution is unstable

**Screening branch:**

$$y_+ = \frac{3\Omega_6}{\Omega_4 a^2} - \frac{3\Omega_3 \Omega_6}{\Omega_4^2 a} + \frac{5}{3} \zeta + \frac{3\Omega_6 \Omega_3^2 + 9\Omega_6^2}{\Omega_4^3} + \mathcal{O}(a)$$

$$y_- = \frac{\zeta}{3} + \frac{4\zeta^2}{27 \Omega_6} \left( \Omega_4 a^2 + \Omega_3 a^3 \right) + \mathcal{O}(a^4)$$

**Notice:** Both screening solutions are stable
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \zeta - 3y + \frac{\Omega_2}{a^2} \right] \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \Omega_2 = 0, \Omega_3 = \Omega_4 = 0 \) and for \( \zeta = 6 \)
Global behavior

$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \zeta - 3y + \frac{\Omega_2}{a^2} \right] \frac{1}{\zeta - y + \frac{\Omega_2}{a^2}}^2$

Solutions $y(a)$ for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = 0$, $\Omega_3 = \Omega_4 = 0$, $\zeta = 0.2$
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \zeta - 3y + \frac{\Omega_2}{a^2} \right] \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \, \Omega_3 = 5, \, \Omega_4 = 0, \, \zeta = 0.2 \). One has \( \Omega_2 = 0 \).
The theory with nonminimal kinetic coupling admits various cosmological solutions.

Ghost-free solutions exist if $\alpha \geq 0$ and $\varepsilon \geq 0$.

The no-ghost conditions eliminate many solutions, as for example the bounces or the “emerging time” solutions.

For $\zeta > \Omega_0$ there exists a ghost-free solution. It describes a universe with the standard late time dynamic dominated by the $\Lambda$-term, radiation and dust. At early times the matter effects are totally screened and the universe expands with a constant Hubble rate determined by $\varepsilon/\alpha$. Since it contains two independent parameters $\zeta$ and $\Omega_0 \sim \Lambda$ in the asymptotics, this solution can have an hierarchy between the Hubble scales at the early and late times. However, at late times it is not screening and dominated by $\Lambda$, thus invoking again the cosmological constant problem.
Conclusions

For $0 < \zeta < \Omega_0$ there exist two ghost-free solutions, A and B. The solution A is sourced by the scalar field, with or without the matter, while the solution B exists only when the matter is present. They both show the screening because their late time behaviour is controlled by $\zeta \sim \varepsilon/\alpha$ and not by $\Lambda$. Therefore, they could in principle describe the late time acceleration while circumventing the cosmological constant problem, and one might probably find arguments justifying that $\varepsilon/\alpha$ should be small. At the same time, these solutions cannot describe the early inflationary phase. Indeed, the near singularity behaviour of the solution B does not correspond to inflation, while the solution A does show an inflationary phase, but with essentially the same Hubble rate as at late times, hence there is no hierarchy between the two Hubble scales.
THANKS FOR YOUR ATTENTION!