On Inflationary Models

Júlio C. Fabris

Departamento de Física - Universidade Federal do Espírito Santo

ICPPA, Moscow, 11 october 2016

The contemporain cosmology

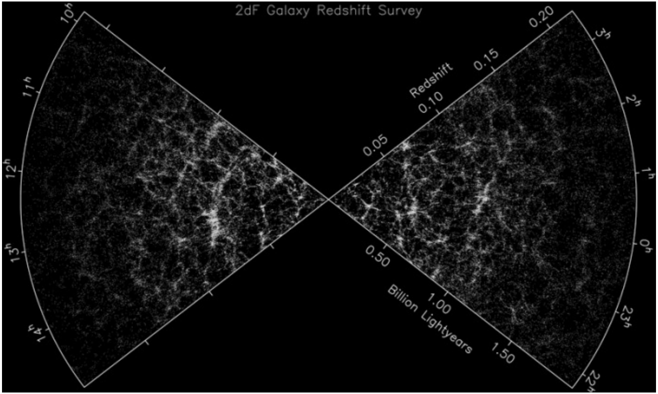
The revolution

- Among the remarkable features of the present researches in cosmology, one can quote:
  1. The precision stage in cosmological researches: An enormous quantity of high quality data makes any theoretical proposal submitted eventually to falsification.
  2. The intime connection between the microphysics, mainly in the very early universe, and the macrophysics at later stages of the evolution of the universe. As an example, quantum effects in the primordial universe determine the distribution of galaxies today in the observed universe.

The contemporain cosmology

The large scale structure

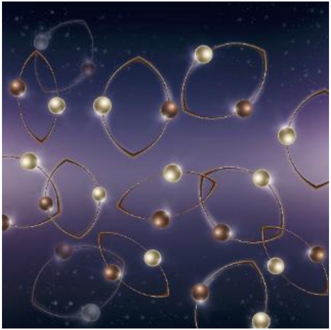
- Hence, the distribution of matter in large scale in the universe,



The contemporain cosmology

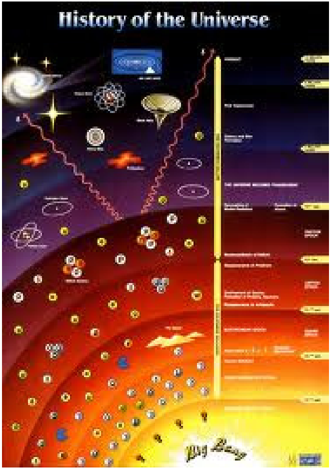
The microphysics

- can be explained by the fluctuations of quantum fields in the very early universe:



The contemporain cosmology

The history of the universe



The inflationary model

The problems of the standard cosmological model

- The Standard cosmological model is very successful in describing the observed universe.
- However, it requires very specific initial conditions.

The inflationary model

The problems of the standard cosmological model

- The isotropy of the cosmic microwave background radiation requires a thermal equilibrium of about 100 regions at the epoch of decoupling between radiation and matter ($z \sim 1.100$), which in principle should not be in causal contact before.
- The density parameter today is around 1, implying a curvature near zero. However, the zero curvature is an unstable point and any departure from in the early universe it implies, in principle, a huge value for the curvature today.
- There is, in the traditional cosmological model, no clear mechanism to generate the perturbation spectrum that must lead to the observed structures in the universe.

The inflationary model

The solution of the problems standard cosmological model

- The solution to all the problems listed above is to suppose that during a given phase in the primordial universe, there was an accelerated expansion phase:

$$\ddot{a} > 0.$$

- From the General Relativity equations applied to an homogeneous and isotropic universe, we have,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$

- Hence, the universe must be dominated by a fluid with negative pressure:

$$p < -\frac{\rho}{3}.$$

The inflationary model

The Inflaton

- The problem that appears immediately with the proposal of an inflationary phase in the primordial universe is the fluid or field responsible for it.
- This component responsible for the inflationary phase is generically called **Inflaton**.

The inflationary model

The Inflaton

- The first proposal was to consider the responsible for the inflationary phase as the vacuum energy.
- In 1965, Gliner has shown that the invariance of the energy-tensor of the vacuum quantum contribution,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu},$$

by Lorentz's transformation (the quantum vacuum must be Lorentz invariant), implies the equation of state,

$$p = -\rho.$$

The inflationary model

The Inflaton

- Using the conservation law,

$$\dot{\rho} + 3H(\rho + p) = 0,$$

it implies,

$$\rho = \text{constant} = \Lambda.$$

- Hence, the vacuum energy is equivalent to a cosmological constant, which satisfy the condition to have inflation.

The inflationary model

The *de Sitter* phase

- The presence of a cosmological constant in the field equations as essentially the unique component of the matter-energy content, induces an accelerated expansion, characterised by the *de Sitter* solution.
- The *de Sitter* solution implies an exponential expansion,

$$a(t) = a_0 e^{Ht}, \quad H = \sqrt{\frac{\Lambda}{3}} = \text{constant}$$

- There is a major problem with this solution: the *de Sitter* space-time is maximal symmetric space, which is invariant by time translation.
- Hence, essentially, in this classical scenario, inflations never ends.

The inflationary model

The Inflaton

- There is an apparent way out of this uncomfortable situation: To consider the vacuum energy not as a cosmological term but as a self-interacting field with a potential with a convenient shape that may lead to *de Sitter* (or a *quasi de-Sitter*) phase, that may end by a suitable quantum mechanism.

The inflationary model

Self-interacting scalar field

- A self-interacting scalar field can be described by the energy-momentum tensor,

$$T_{\mu\nu} = \phi_{;\mu}\phi_{\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho} + g_{\mu\nu}V(\phi).$$

The inflationary model

Self-interacting scalar field

- In a FLRW background, we can identify the energy density and pressure:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

The inflationary model

Self-interacting scalar field

- The equations of motion are:

$$\begin{aligned} 3H^2 &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \\ \ddot{\phi} + 3H\dot{\phi} &= -V_\phi(\phi). \end{aligned}$$

The inflationary model

Self-interacting scalar field

- The condition for inflation are given by the slow-roll parameters,

$$\epsilon = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2,$$
$$\eta = M_{Pl}^2\frac{V''}{V},$$

which must be much smaller than one in order to erase the sensibility to the initial conditions.

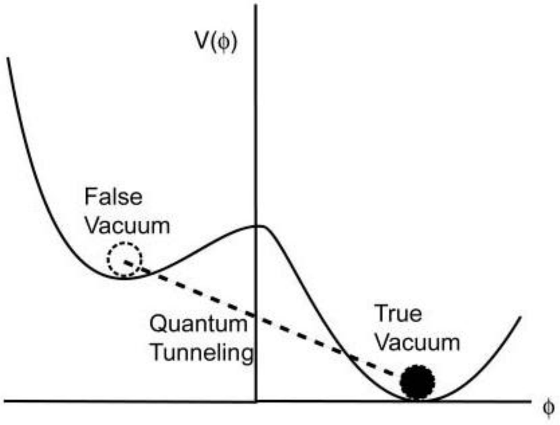
The inflationary model

The different models of inflation

- The different models for the inflation field can be divided as,
  1. Old inflation.
  2. Large field inflation.
  3. Small field inflation.

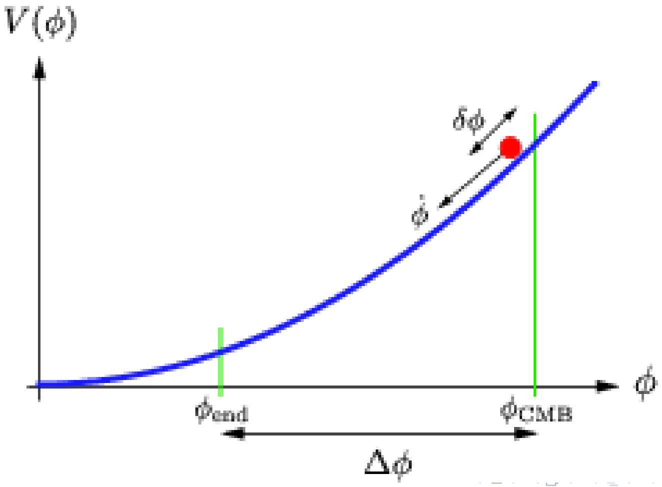
The inflationary model

The Inflaton - old inflation

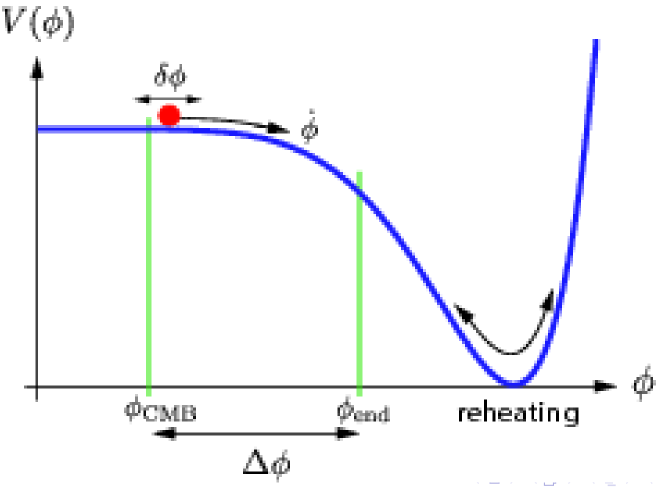


The inflationary model

Large field models



The inflationary model
Small field models



The inflationary model

Observational constraint

- The spectral index of the scalar and tensorial perturbations are directly related to the slow-row parameters:

$$\begin{aligned} n_s &= 1 - 6\epsilon + 2\eta, \\ n_T &= -2\epsilon. \end{aligned}$$

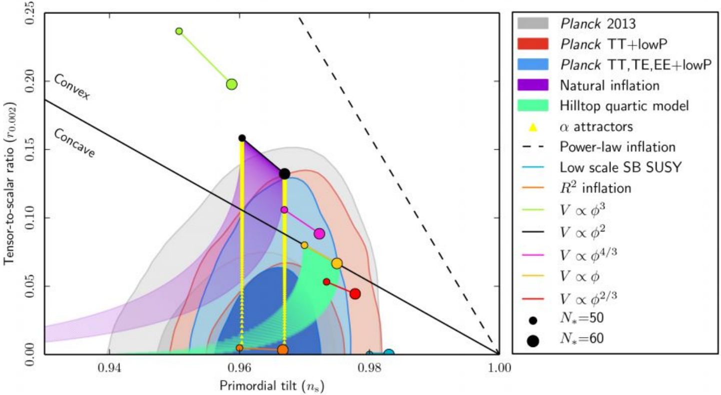
The inflationary model

Observational constraint

- There are other important parameters.
  1. The e-fold N at the end of inflation
  $$N = \ln \frac{a_f}{a_i},$$
  2. and the ratio between the gravitational waves and scalar modes due to inflation,
  $$r \propto \epsilon.$$
  (In general, a huge quantity of primordial gravitational waves is produced during inflation).

The inflationary model

Observational constraint



$f(R)$ theories

The action

- The $f(R)$ theory is described by the following action:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) \; ,$$

where $\kappa^2 = 8\pi G$.

- Such theories can be mapped into scalar-tensors theories with a potential.

$f(R)$ theory
$f(R)$ theories

- In fact, all theories of the type,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) \ ,$$

can be recast in a scalar-tensor form,

$$S = \int dx^4\sqrt{-g}\phi[R + 2V(\phi)],$$

with

$$\phi = f'(R) \quad , \quad V(\phi) = \frac{1}{2}\left(\frac{f'}{f} - R\right).$$

$f(R)$ theory

The dynamical system for $f(R)$

- Assuming a flat FLRW metric:

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j \ ,$$

we can easily cast the evolution equation for $f(R)$ gravity as a dynamical system formed by the following equations:

$$6f'H^2 = Rf' - f - 6H\dot{R}f'' \ ,$$
$$R = 6(\dot{H} + 2H^2) \ ,$$

where $H \equiv \dot{a}/a$, the dot denotes derivation with respect to the cosmic time and the prime denotes derivation with respect to R.

$f(R)$ theory

The dynamical system for $f(R)$

- We find the following autonomous system:

$$\begin{cases} \dot{R} = \frac{1}{6Hf''}\left[Rf' - f - 6f'H^2\right] \ , \\ \dot{H} = \frac{R}{6} - 2H^2 \ , \end{cases}$$

where we assumed f'' and H different from zero.

- The critical points of the above autonomous system are given by

$$\begin{cases} Rf' - f - 6f'H^2 = 0 \ , \\ R = 12H^2 \ . \end{cases}$$

$f(R)$ theory

The dynamical system for $f(R)$

- Substituting the second equation in the first one, one obtains the following ordinary differential equation for f:

$$\frac{f'}{f} = \frac{2}{R} ,$$

which has the solution

$$f(R) = \alpha R^2 ,$$

where α is an integration constant.

$f(R)$ theory

The dynamical system for $f(R)$

- Rewriting the system for $f(R) = \alpha R^2$, one obtains

$$\begin{cases} \dot{R} = \frac{R}{12H}\left[R - 12H^2\right] \ , \\ \dot{H} = \frac{R}{6} - 2H^2 \ . \end{cases}$$

$f(R)$ theory

The dynamical system for $f(R)$

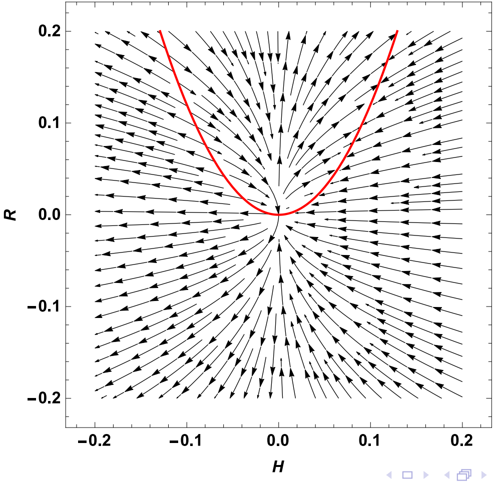
- From the dynamical system it can be verified that inflation can happen at any energy scale and they lie on the parabola $R = 12H^2$ and they are stable points.
- Linearising the dynamical system about a critical point $R_0 = 12H_0^2$, we obtain:

$$\begin{cases} \dot{\epsilon} = H_0\epsilon - 24H_0^2\eta \ , \\ \dot{\eta} = \frac{\epsilon}{6} - 4H_0\eta \ . \end{cases}$$

- The secular equation for the above system matrix is $\lambda(\lambda + 3H_0) = 0$.
- Therefore, one eigenvalue is vanishing and the other is negative (for $H_0 > 0$, which is the region of our interest).
- This means that the critical points are attractors.

$f(R)$ theory

The dynamical system for $f(R)$



Starobinsky's model
Dynamical system

- The Starobinsky's model $f(R) = R + \alpha R^2$, on the other hand, tells a different story.
- The dynamical system becomes:

$$\begin{cases} \dot{R} = \frac{1}{12H\alpha}\left[\alpha R^2 - 6H^2(1 + 2\alpha R)\right] \ , \\ \dot{H} = \frac{R}{6} - 2H^2 \ , \end{cases}$$

in which α can be arranged as follows:

$$\begin{cases} \alpha^{3/2}\dot{R} = \frac{1}{12H\sqrt{\alpha}}\left[\alpha^2 R^2 - 6\alpha H^2(1 + 2\alpha R)\right] \ , \\ \alpha\dot{H} = \frac{\alpha R}{6} - 2\alpha H^2 \ , \end{cases}$$

in order for the dimensionless quantities αR, αH^2 and $t/\sqrt{\alpha}$ to appear.

Starobinsky's model

Dynamical system

- Looking for critical points at finite distance from the origin in the phase-space (H, R), we put $\dot{H} = 0$, obtaining thus $R = 12H^2$ implying:

$$\begin{cases} \dot{R} = -\frac{H}{2\alpha} \ , \\ \dot{H} = 0 \ . \end{cases}$$

- Therefore, we conclude that $R = 0 = H$, i.e. Minkowski space, is a critical point.

Starobinsky's model

Dynamical system

- In Starobinsky's model there are also two critical points at infinity.
- One of these represents a de Sitter repulsor and thus may provide a transient phase of inflation at high-energy scales, i.e. for $\alpha R \gg 1$.

Starobinsky's model

Dynamical system

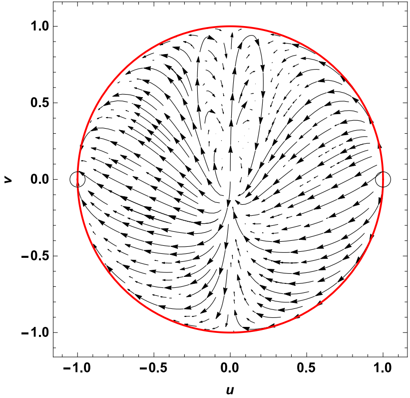


Figure : Equatorial plane of the Poincaré's sphere for the $f(R) = R + \alpha R^2$ model.

An exponential correction to Starobinsky's model

Work in progress with Oliver Piattella and Tays Miranda

- Since Starobinsky's model is very successful, it is interesting to investigate small deviations from it in order to test its robustness.
- What we learned from the dynamical system analysis of the previous section is that $f(R)$ should be quadratic only for large values of R. Therefore, we propose the following model:

$$f(R) = R + \alpha R^2 - 2\Lambda e^{-\alpha R} \ ,$$

  i.e. an exponential correction of Starobinsky's model.
- For $\alpha R \gg 1$ our model reproduces the successful inflationary paradigm of Starobinsky's model and when $\alpha R \ll 1$, the above function $f(R)$ can be approximated as

$$f(R) \sim R - 2\Lambda \ .$$

An exponential correction to Starobinsky's model

The model

- The dynamical system becomes:

$$\begin{cases} \dot{R} = \frac{1}{12H\alpha}\frac{\alpha R^2 + 2\Lambda(\alpha R + 1)e^{-\alpha R} - 6H^2 - 12H^2\alpha R - 12H^2\alpha\Lambda e^{-\alpha R}}{1 - \alpha\Lambda e^{-\alpha R}} , \\ \dot{H} = \frac{R}{6} - 2H^2 . \end{cases}$$

An exponential correction to Starobinsky's model

The model

- Looking for critical points, we get the same ones as in Starobinsky's model at infinity, representing inflation, and another one given by the following transcendental equation:

$$R = 2\Lambda(\alpha R + 2)e^{-\alpha R} .$$

- For $\alpha R \ll 1$, the above equation can be solved approximately as:

$$R \approx \frac{4\Lambda}{1 + 2\alpha\Lambda} .$$

An exponential correction to Starobinsky's model

The model

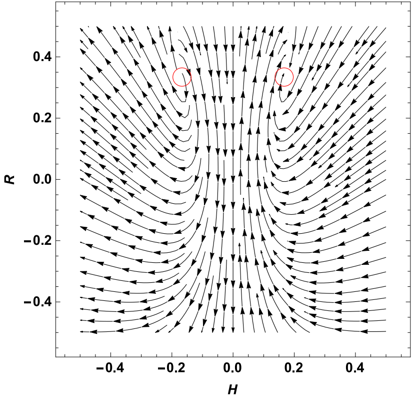


Figure : Phase space diagram for the $f(R) = R + \alpha R^2 - 2\Lambda e^{-\alpha R}$ model.

An exponential correction to Starobinsky's model

The model

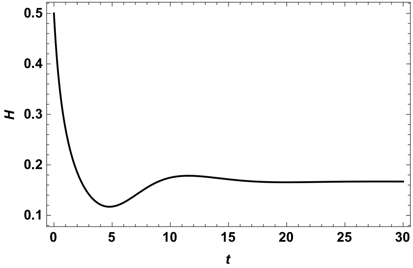


Figure : Evolution of $H(t)$ computed from system (1). We chose initial conditions $H_i = 0.5$ and $R_i = 0.4$ and fixed the parameter $\Lambda = 0.1$.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

The action

- A given $f(R)$ theory can be mapped into a scalar field theory in the Jordan frame.
- The action,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) \, ,$$

can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}\phi R - V(\phi)\right] \, ,$$

where

$$\phi \equiv \frac{1}{\kappa}\frac{df}{dR} \, , \qquad V(\phi) \equiv \frac{1}{2\kappa^2}\left(R\frac{df}{dR} - f\right) \, .$$

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Conformal transformation

- Upon the conformal transformation $\hat{g}_{\mu\nu} = \kappa\phi g_{\mu\nu}$, we can write the above action in the Einstein frame as follows:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi)\right] ,$$

where

$$U \equiv \frac{V}{\kappa^2\phi^2} , \qquad \chi \equiv \sqrt{\frac{3}{2}}\frac{1}{\kappa}\ln\kappa\phi .$$

- For the exponential model (1), the above defined quantities take the following form:

$$\kappa\phi = 1 + 2\alpha R + 2\alpha\Lambda e^{-\alpha R} , \qquad 2\kappa^2 V = \alpha R^2 + 2\Lambda(\alpha R + 1)e^{-\alpha R} .$$

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Conformal transformation

- Finding an explicit form $V(\phi)$ for the potential is then complicated by the exponential function.
- Formally, using the Lambert W function (or product logarithm), we can write:

$$2\alpha R = \kappa\phi - 1 + 2W\left[-\alpha\Lambda e^{(1-\kappa\phi)/2}\right] .$$

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Conformal transformation

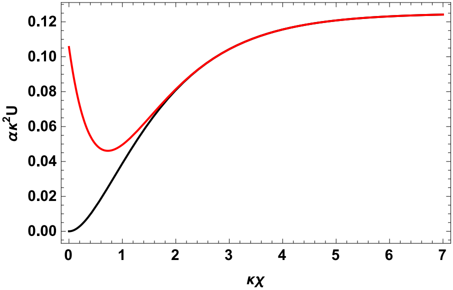


Figure : Evolution of the effective potential $U(\chi)$. The black solid line represents Starobinsky's model (i.e. $\Lambda = 0$), whereas the red-line is its exponential extension for $\alpha\Lambda = 0.1$.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Conformal transformation

- The more intriguing feature of the potential of our exponential model, compared with the Starobinsky's model's one , is the presence of a minimum for a non-vanishing value of the scalar field.
- This suggests that the reheating phase of the Universe takes place earlier, if compared with the Starobinsky's model.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters

- Let us now investigate the slow-roll parameters. They can be defined through the potential $U(\chi)$ as follows:

$$\epsilon_U = \frac{1}{2\kappa^2}\left(\frac{U_{,\chi}}{U}\right)^2 , \qquad \eta_U = \frac{1}{\kappa^2}\frac{U_{,\chi\chi}}{U} .$$

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters

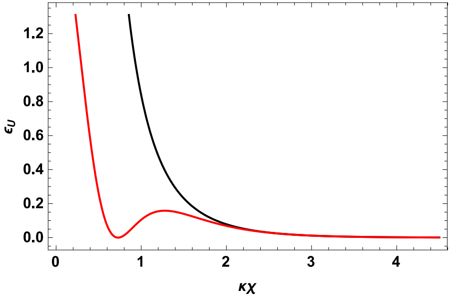


Figure : Evolution of the slow-roll parameter ϵ_U. The black solid line represents Starobinsky's model (i.e. $\Lambda = 0$), whereas the red-line is its exponential extension for $\alpha\Lambda = 0.1$.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters

- The end of the inflationary period, i.e. when $\epsilon_U \approx 1$ takes place for smaller values of the field χ in the exponential extension of Starobinsky's model.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters

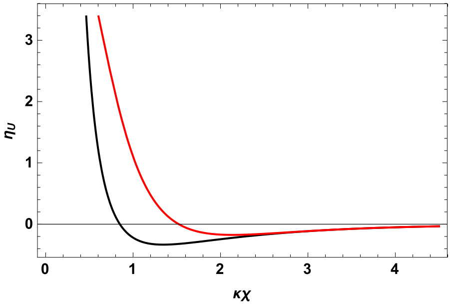


Figure : Evolution of the slow-roll parameter η_U. The black solid line represents Starobinsky's model (i.e. $\Lambda = 0$), whereas the red-line is its exponential extension for $\alpha\Lambda = 0.1$.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters

- The number of e-foldings is defined as:

$$N \equiv \kappa^2 \int_{\chi_f}^{\chi} \frac{U}{U_{,\chi}} d\chi \ ,$$

where χ_f is the scalar field at which inflation ends. Usually, that is determined by the condition $\epsilon_U(\chi_f) = 1$.

- The observables quantities are:

$$n_s = 1 - 6\epsilon_U + 2\eta_U \ , \qquad r = -8n_t \ , \qquad n_t = -2\epsilon_U \ ,$$

where n_s is the scalar spectral index, n_t is the tensor spectral index and r is the tensor-to-scalar power ratio.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters

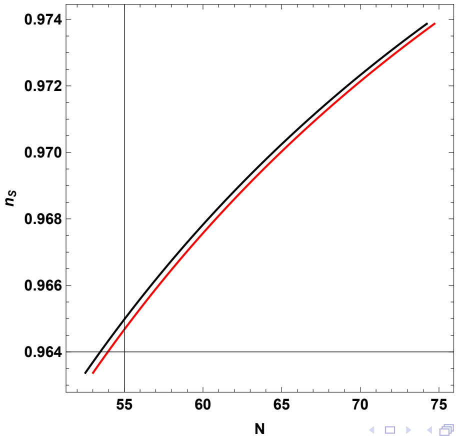
- The Planck collaboration put the following constraints on these quantities:

$$n_s = 0.968 \pm 0.006 \,, \qquad r < 0.12 \,,$$

the former being at 68% CL and the latter at 95% CL.

Inflation and slow-roll parameters in the exponential variation of Starobinsky's model

Slow-roll parameters



Conclusions

Conclusions

- Inflation seems to be a necessary ingredient to give sense to the Standard Cosmological Model.
- It has a huge quantity of different implementations of inflationary scenarios.
- The Starobinsky model seems to be one of the most successful one.
- It gives the correct observational estimations, but it predicts almost no production of gravitational waves.
- Some variations around the Starobinsky model conducts to essentially the same results.