On multidimensional gravity and the Casimir effect

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Motivation

- The idea of extra dimensions is a powerful methodological framework in modern theoretical physics (Kaluza–Klein theories, superstrings, brane worlds, etc.).
- There are a number of multidimensional (including cosmological) models such as $F(R)$-theories and others. Among them, it is interesting to search for realistic models explaining the unobservable nature of extra dimensions, possible mechanisms of their stabilization, and providing the agreement with the realistic cosmological scenarios and observations.
- Due to small compact extra dimensions, there can be possible quantum effects such as the topological Casimir effect

The goals

- We study the properties of an effective potential for the scale factor of extra dimensions in a Kaluza-Klein-type model with curvature-nonlinear terms and a spherical extra space.
- We take into account the topological Casimir energy of scalar KK modes
- We demonstrate the existence of physically appropriate minima of the potential, providing stabilization of compact extra dimensions in agreement with the observed accelerated expansion of the Universe.

The model

Consider a Kaluza-Klein-type model with curvature-nonlinear terms and a spherical extra space.

- Manifold: $\mathcal{M}^D = \mathbb{M}^4 \times \mathbb{S}^n, \quad D = 4 + n$
- Action:

$$S = \frac{1}{2} m_D^{D-2} \int \sqrt{g_D}\, d^D x \left[F(R) + c_1 R^{AB} R_{AB} + c_2 R^{ABCD} R_{ABCD} \right]$$

$$m_D \equiv 1/r_0, \quad g_D = |\det(g_{MN})|$$

We neglect off-diagonal (gauge) KK modes and analyse the ground state with the geometry $\mathbb{M}^4 \times \mathbb{S}^n$ and the metric:

- $ds^2 = g_{\mu\nu} dx^\mu dx^\nu - e^{2\beta(x^\mu)} d\Omega_n^2$,
  where the scale factor of extra space is governed by a 4D scalar field $\beta(x)$.

The methodology

(a) Integrate over the sphere $\mathbb{S}^n$ thus reducing all quantities to 4D variables and $\beta(x^\mu)$. In particular,

$$R = R_4 + \phi + 2n\Box\beta + n(n+1)\left(g^{\mu\nu}\partial_\mu\beta\partial_\nu\beta\right)$$

$$\phi(x^\mu) \equiv m_D^2\, n(n-1)\ \mathrm{e}^{-2\beta(x^\mu)}$$

(b) Suppose that all quantities are slowly varying as compared with the D-dimensional Planck scale, i.e., consider each derivative ∂_μ as an expression with a small parameter ε and neglect all quantities of orders higher than $O(\varepsilon^2)$.

(c) Introduce the Casimir energy density of a scalar field in $\mathbb{M}^4 \times \mathbb{S}^n$ having the form $\mathcal{E}_{\rm Cas} \sim C_n r^{-4}$, where r is the radius of the extra factor space.

(d) Transition from the Jordan conformal frame to the Einstein frame where the field ϕ is minimally coupled to the 4D curvature.

The reduced action

- The reduced action in the Jordan frame:

$$S = \frac{1}{2}\mathcal{V} m_D^2 \int \sqrt{g_4} d^4x \Big[\mathrm{e}^{n\beta} F'(\phi) R_4 + [\mathrm{Kin}]_J - 2(V_J + V_{\mathrm{Cas}}) \Big].$$

- Here:

$$F'(\phi) \equiv dF/d\phi, \quad \mathcal{V} = 2\pi^{(n+1)/2}/\Gamma(\tfrac{1}{2}(n+1)) = \mathrm{Vol}(S^n)$$

$$[\mathrm{Kin}]_\mathrm{J} = (\partial\beta)^2 \,\mathrm{e}^{n\beta} \big[n(n-1)(4\phi F'' - F') + 4(c_1 + c_2)\phi \big]$$

$$V_J(\phi) = -\frac{1}{2}\,\mathrm{e}^{n\beta}[F(\phi) + c_J\,\mathrm{e}^{-4\beta}], \quad V_{\mathrm{Cas}} = C_n r_0^{-2} \mathcal{V}^{-1}\,\mathrm{e}^{-4\beta}$$

$$c_J \equiv n(n-1) r_0^{-4}[(n-1)c_1 + 2c_2]$$

- C_n are numerical coefficients defining the Casimir energy density for odd number of extra dimensions (Candelas, Weinberg, 1984; Milton, 2001).

The Einstein frame

- Transition to the Einstein frame via the conformal mapping:

$$g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu} = |f(\phi)| g_{\mu\nu}, \quad f(\phi) = \mathrm{e}^{n\beta} F'(\phi)$$

- The reduced 4D action:

$$S = \frac{1}{2}\mathcal{V}[n] m_D^2 \int \sqrt{\widetilde{g}} \left[\widetilde{R}_4 + [\mathrm{Kin}]_{\mathrm{E}} - 2V_{\mathrm{E(tot)}}(\phi) \right]$$

$$[\mathrm{Kin}]_{\mathrm{E}} = (\partial\beta)^2 \left[6\phi^2 \left(\frac{F''}{F'} \right)^2 - 2n\phi \frac{F''}{F'} + \frac{1}{2} n(n+2) + \frac{4(c_1+c_2)\phi}{F'} \right]$$

$$\boxed{V_{\mathrm{E(tot)}}(\phi) = \frac{\mathrm{e}^{-n\beta}}{2F'(\phi)^2} \left[-F(\phi) - c_J \mathrm{e}^{-4\beta} + C_n r_0^{-2} \mathcal{V}^{-1} \mathrm{e}^{-(n+4)\beta} \right]}$$

Physical constraints

- We describe our space-time classically, therefore the size $r = r_0\,\mathrm{e}^{\beta}$ of the extra dimensions should exceed the fundamental length scale $r_0 = 1/m_D$, i.e., $\mathrm{e}^{\beta} \gg 1$.
- The extra dimensions should not be directly observable, which means that $r = r_0\,\mathrm{e}^{\beta} \lesssim 10^{-17}$ cm, associated with the TeV energy scale.
- The effective cosmological constant Λ_{eff}, corresponding to the value of $V_{\text{E(tot)}}$ at its minimum, should conform to observations, which means that

$$\Lambda_{\text{eff}} > 0 \quad \text{but} \quad \Lambda_{\text{eff}}/m_4^2 \sim 10^{-120},$$

  where $m_4 \sim 10^{-5}$ g is the familiar 4D Planck mass.
- Non-phantom character of the scalar field: $[\mathrm{Kin}]_{\mathrm{E}}(\phi) > 0$

The specific effective potential

- Specify the function $F(R)$ in the general quadratic form:

$$F(R) = -2\Lambda_D + F_1 R + F_2 R^2, \quad \Lambda_D,\ F_1,\ F_2 = \text{const}$$

- Denoting $x = \mathrm{e}^{-\beta}$, introduce the dimensionless potential

$$W(x) \equiv r_0^2 V_{\mathrm{E(tot)}}(\phi) = \frac{\lambda x^n - k_1 x^{n+2} - k_2 x^{n+4} + k_3 x^{2n+4}}{r_0^2[F_1 + 2n(n-1)r_0^{-2}F_2]^2},$$

where

$$\lambda = r_0^2 \Lambda_D, \quad k_1 = n(n-1)/2,$$

$$2k_2 = n^2(n-1)^2 r_0^{-2} F_2 + r_0^2 c_J, \quad k_3 = C_n/\mathcal{V}.$$

Possible stationary case

- We search for small stable positive minima of $W(x)$ near the point $x_{\min} \lesssim 0.1$ parametrized by choosing the value of λ.
- $r(x_{\min}) = r_0/x_{\min} = \sqrt{\mathcal{V}(n)}/(m_4 x_{\min}) \sim 10^{-31}\mathrm{cm}$
- To demonstrate the existence of such minima under our physical assumptions, consider the following simple set of parameters:

$$F_1 = 1, \quad F_2 = 0 \quad \Longrightarrow \quad F(R) = R - 2\Lambda_D$$

$$n = 3, \quad C_3 = 7.5687046 \times 10^{-5}, \quad \mathcal{V} = 2\pi^2,$$

$$k_2 = -200, \quad k_3 \simeq 10^{-4}$$

Plots of the potential

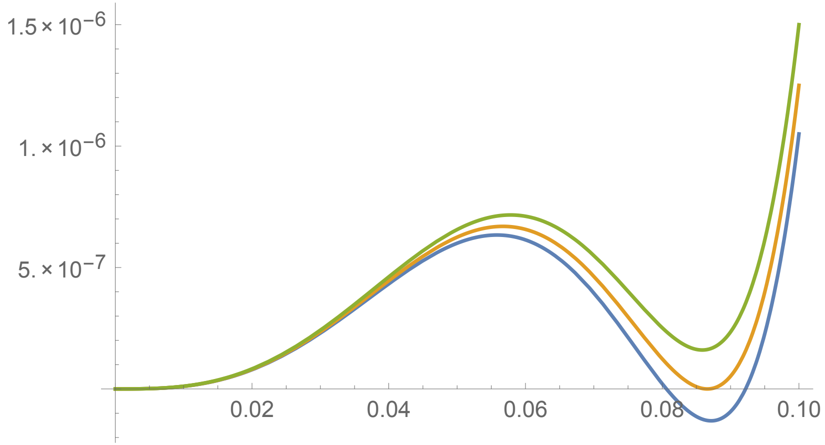

Figure: Plots of $W(x)$ for $n = 3$, $F_2 = 0$, $k_2 = -200$, $k_3 = 10^{-4}$, and $\lambda = 0.01105,\ 0.01125,\ 0.0115$ (bottom-up)

Plots of the potential

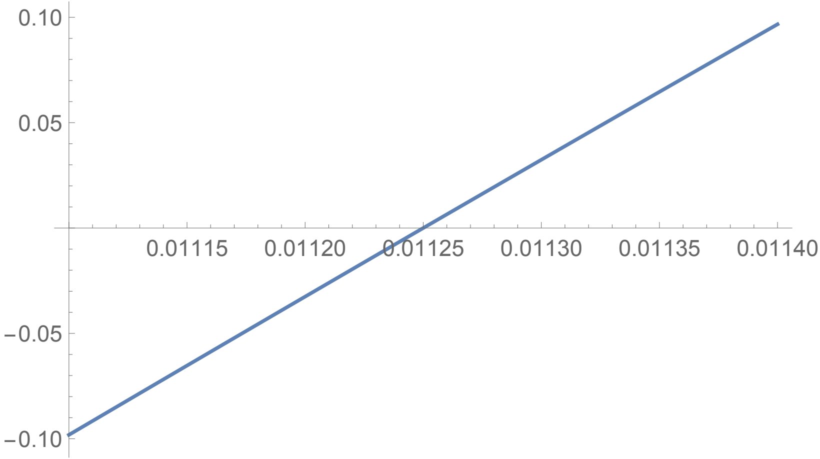

Figure: Plot of $W(x_{\min})$ as a function of λ for the same values of $n,\ F_2,\ k_2,\ k_3$

Discussion

- In the framework of curvature-nonlinear multidimensional gravity, we have proved the existence of minima of the effective potential, corresponding to a stable stationary size of extra dimensions.
- The existence of these minima is provided by a Casimir contribution to the potential combined with terms quadratic in the components of the curvature tensor.
- The effective cosmological constant $\Lambda_{\rm eff} = W(x_{\min})/r_0^2 = W(x_{\min}) m_4^2/\mathcal{V}(n)$. The quantity $W(x_{\min})$ should insignificantly exceeds the number 10^{-120}. It is achieved by appropriate choosing the parameters, for example, at $\lambda \approx 0.01125$ in the figure above.
- The Casimir contribution ($\sim k_3 x_{\min}^{10} \gg 10^{-120}$) is in general much larger than a cosmological energy density. In realistic models of this type there can be a fine-tuned compensation of the Casimir energy by nonlinear curvature terms.

Discussion

Possible extensions:

- Small perturbations of stationary state and the variations of fundamental constants
- More accurate analysis of the Casimir contribution (massive fields, gauge and gravitational modes, ...)
- Another geometries of extra factor spaces
- Analysis of the early inflationary epoch with possible quantum tunneling between minima obtained

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**Thank you
for your attention!**