Wormholes leading to extra dimensions

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Motivation and plan

- Multidimensional theories: a great variety of geometries, topologies and compactification schemes. Extra dimensions can become large and observable in some part of space $\Rightarrow$ another physics.
- A way to study such possible situations: a search for the corresponding solutions of multidimensional Einstein equations.
- We consider 6D manifolds $\mathbb{M}_0 \times \mathbb{M}_1 \times \mathbb{M}_2$; $\mathbb{M}_0$: 2D Lorentzian; each of $\mathbb{M}_1$, $\mathbb{M}_2$: either $\mathbb{S}^2$ (2-sphere) or $\mathbb{T}^2$ (2-torus). Source of gravity: a minimally coupled (phantom) scalar field.
- We select the possible asymptotic behaviors of the metric functions compatible with the field equations. Their choice is rather narrow.
- Two examples of wormhole solutions with the expected properties. Our "end": our 4D space $\times \mathbb{T}^2$ (small); "far end": large $\mathbb{T}^2$.
- Example 1: a massless scalar field (based on a well-known more general solution). "Far end": extra $\mathbb{T}^2$ has a large constant size.
- Example 2: a nonzero potential $V(\phi)$; "far end": 6D AdS, all spatial dimensions are infinite.

Basic equations

Action: $S = \frac{m_6^2}{2} \int \sqrt{|g_6|} \Big[R_6 + 2\varepsilon_\phi g^{AB} \partial_A \phi \partial_B \phi - 2V(\phi) \Big]$,

where: m_6 = 6D Planck mass, R_6 = 6D Ricci scalar, $g_6 = \det(g_{AB})$,
$\varepsilon_\phi = 1$ for a normal, canonical scalar field, $\varepsilon_\phi = -1$ for a phantom one;
$V(\phi)$ = scalar field potential; $A, B, \ldots = \overline{0, 5}$.

Equations: $2\varepsilon_\phi \Box_6 \phi + dV/d\phi = 0$

$$R^A_B = -\widetilde{T}^A_B \equiv -T^A_B - \tfrac{1}{4}\delta^A_B T^C_C \equiv -2\varepsilon_\phi \partial^A \phi \partial_B \phi + \tfrac{1}{2} V(\phi) \delta^A_B,$$

R^A_B = 6D Ricci tensor, T^A_B = stress-energy tensor (SET) of ϕ.

Metric: $ds^2 = A(x)dt^2 - \dfrac{du^2}{A(x)} - R(x)d\Omega_1^2 - P(x)d\Omega_2^2$,

x = "radial" coordinate:
$d\Omega_1^2, d\Omega_2^2$ = x-independent metrics on 2D manifolds of unit size;
$R(x) = r^2(x)$ = size of $\mathbb{M}_1$ (2-sphere or 2-torus);
$P(x) = p^2(x)$ = size of $\mathbb{M}_2$ (2-sphere or 2-torus);
Scalar field: $\phi = \phi(x)$.

Basic equations — 2

Which of $\mathbb{M}_{1,2}$ belongs to our 4D space-time and which is "extra"? Everything depends on their size.

SET:

$$\widetilde{T}^t_t = \widetilde{T}^2_2 = \widetilde{T}^3_3 = \widetilde{T}^4_4 = \widetilde{T}^5_5 = -\tfrac{1}{2}V(\phi),$$
$$\widetilde{T}^t_t - \widetilde{T}^u_u = 2\varepsilon_\phi A(x)\phi'^2.$$

Symmetry of the problem $\Rightarrow$ 4 independent equations (prime $= d/dx$):

$$R^t_t = -\widetilde{T}^t_t \quad \Rightarrow \quad -\frac{1}{PR}(A'PR)' = V(\phi), \qquad (1)$$

$$R^t_t - R^x_x = -\widetilde{T}^t_t + \widetilde{T}^x_x \quad \Rightarrow \quad r''/r + p''/p = -\varepsilon_\phi \phi'^2, \qquad (2)$$

$$R^t_t - R^2_2 = 0 \quad \Rightarrow \quad [P(AR' - A'R)]' = 2\varepsilon_1 P, \qquad (3)$$

$$R^t_t - R^4_4 = 0 \quad \Rightarrow \quad [R(AP' - A'P)]' = 2\varepsilon_2 R. \qquad (4)$$

($\varepsilon_1 = 1 \Leftrightarrow \mathbb{M}_1 =$ sphere, $\varepsilon_1 = 0 \Leftrightarrow \mathbb{M}_1 =$ torus. The same for ε_2.)

Note: In Eqs. (3) and (4) — only metric functions! 2 eqs for 3 unknowns. If we know $A(x), R(x), P(x)$, we find $V(x)$ and $\phi(x)$ from (1) and (2).

Types of geometries

Eq. (2) $\Rightarrow$ solutions with $r > 0$ and $p > 0$ in the whole range $x \in \mathbb{R}$ exist only with $\varepsilon_\phi = -1$, i.e., a phantom field, since they require $r'' > 0$ and $p'' > 0$.

- **SS (double spherical) space-times:** $\varepsilon_1 = \varepsilon_2 = 1$. If spheres $\mathbb{M}_1$ are large and $\mathbb{M}_2$ are small (or vice versa), there is static spherical symmetry in our space-time and a spherical extra space.
  Both spheres are large $\Rightarrow$ 6D space-time, all dimensions are observable.
- **ST (spherical-toroidal) space-times:** the case $\varepsilon_1 = 1,\ \varepsilon_2 = 0$ (or vice versa). If $\mathbb{M}_1$ is large and $\mathbb{M}_2$ small, we have static spherical symmetry in our space-time and a toroidal extra space. The opposite situation is also possible as well as a total observable 6D geometry.
- **TT (double toroidal) space-times:** if $\varepsilon_1 = \varepsilon_2 = 0$, we have the same as before but both $\mathbb{M}_1$ and $\mathbb{M}_2$ are toroidal.

Our interest: finding configurations where $x \in \mathbb{R}$ and there are different asymptotic behaviors of $R = r^2$ and $P = p^2$ as $x \to \pm\infty$.

Possible asymptotic behavior

We check which kinds of asymptotics (4D or 6D flat, dS, AdS) are admitted by the pure metric equations (3) and (4):

$$[P(AR' - A'R)]' = 2\varepsilon_1 P \quad (3), \qquad [R(AP' - A'P)]' = 2\varepsilon_2 R. \quad (4)$$

Example of asymptotic analysis. Consider an asymptotically flat 4D spherically symmetric space-time with constant spherical extra dimensions. This means $\varepsilon_1 = \varepsilon_2 = 1$ and, without loss of generality ($\mathrm{fin} \equiv \mathrm{const} > 0$),

$$A(x) \to \mathrm{fin}, \qquad R(x) \sim x^2, \qquad P(x) \to \mathrm{fin} \qquad (5)$$

as $x \to \infty$. We substitute to (3) and (4) the expansions

$$A(x) = A_0 + \frac{A_1}{x} + \ldots, \qquad R(x) = x^2(1 + o(1)), \qquad P(x) = P_0 + \frac{P_1}{x} + \ldots,$$

so that $R' \sim x$, $A' \sim x^{-2}$ or even smaller, and the l.h.s. of (3) tends, in general, to a nonzero constant, which agrees with $P \to \mathrm{fin}$ on the r.h.s..
However, in (4) the expression in square brackets tends to a constant, hence its derivative vanishes, while the r.h.s., equal to $2R$, should behave as x^2.
Thus the conditions (5) are incompatible with the field equations.
The same follows if consider $x \to -\infty$ and/or exchange $R(x)$ and $P(x)$.

Possible asymptotic behavior — 2

In the above manner we analyze different opportunities and obtain the table:

No.	Asymptotic behavior			6D geometries			Comments
	$A(x)$	$R(x)$	$P(x)$	SS	ST	TT	
0	fin	fin	fin	$-$	$-$	$\pm$	$\mathbb{M}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$
1	fin	fin	x^2	$-$	$-$	$-$	none
2	fin	x^2	fin	$-$	$+$	$-$	$\mathbb{M}^4 \times \mathbb{T}^2$
3	fin	x^2	x^2	$+$	$-$	$-$	$\mathbb{M}^6$
4	x^2	fin	fin	$+$	$-$	$-$	$\mathrm{dS}_2 \times \mathbb{S}^2 \times \mathbb{S}^2$
5	x^2	fin	x^2	$\pm$	$\pm$	$-$	$\mathrm{dS}_4 \times \mathbb{S}^2$
6	x^2	x^2	fin	$\pm$	$-$	$-$	$\mathrm{dS}_4 \times \mathbb{S}^2$
7	x^2	x^2	x^2	$\pm$	$\pm$	$\pm$	(A)dS_6

Here: $+$ (plus) means "possible", $-$ (minus) — "impossible",
$\pm$ — possible under special conditions on the parameters.
$\mathbb{M}$ stands for Minkowski; the comment "dS" means a de Sitter asymptotic with $A(x) \sim -x^2$, whereas an AdS behavior ($A \sim x^2$) is impossible.

In particular, wormholes with $\mathbb{M}^4 \times \mathbb{T}^2$ on one or both ends are only possible with **ST** geometry. Further on: two examples of such wormholes.

Example 1: ST wormholes with a massless scalar

It is a special case of multidimensional solutions with a massless scalar known for a long time (KB, 1995, KB, Ivashchuk and Melnikov, 1997, etc.)

Metric and scalar field:

$$ds^2 = dt^2 - \mathrm{e}^{-4nu}\left[dz^2 + (z^2 + \overline{k}^2)d\Omega_1^2\right] - \mathrm{e}^{2nu} d\Omega_2^2,$$

$$\phi = Cu, \qquad u := \frac{1}{\overline{k}} \cot^{-1}\left(\frac{-z}{\overline{k}}\right), \tag{6}$$

Here, $\overline{k}$, n, C = integration constants, such that $\overline{k}^2 + 3n^2 = 2C^2$.
It is a **spherically symmetric, twice asymptotically flat wormhole in 4D subspace** $\mathbb{M}_0 \times \mathbb{M}_1$ ($M_1 = \mathbb{S}^2$) with a toroidal extra space $\mathbb{M}_2 = \mathbb{T}^2$.

Note: z is another coordinate than x used in other parts of this presentation.

Example 1 — continued

Size of $\mathbb{T}^2$: $p = p_- = 1 \quad (z = -\infty)$ — "here",
$p = p_+ = \mathrm{e}^{n\pi/\bar{k}} p_- \quad (z = +\infty)$ — at the "far end".

Wormhole throat: a minimum of $r(z) = \mathrm{e}^{-2nu}(z^2 + \bar{k}^2)^{1/2}$, located at $z = 2n$, its radius:

$$r_{\min} = \sqrt{\bar{k}^2 + 4n^2} \exp\left(\frac{2n}{\bar{k}} \cot^{-1} \frac{2n}{\bar{k}}\right). \quad (7)$$

Suppose that the size of extra dimensions p_- on "our" end, $z = -\infty$, is small enough to be invisible by modern instruments, say,

$$p_- = 10^{-17} \text{ cm}.$$

On the other end, it depends on the ratio $n/\bar{k}$.
Thus, to obtain $p = p_+ \sim 1$ m, we should take $n/\bar{k} \approx 14$.

The throat radius also depends on n and $\bar{k}$. It is not too large if they take modest values. Thus, for $n/\bar{k} = 14$, we have $r_{\min} \approx 76\bar{k}p_-$. To obtain a large enough throat for passing of a macroscopic body, say, $r_{\min} = 10$ meters, one has to suppose $\bar{k} \sim 10^{18}$.

Example 2: ST asymptotically AdS wormholes

With nonzero potentials $V(\phi)$, in most cases solutions can be found only numerically, with one exception (recall that $\varepsilon_2 = 0$ in the ST case!):

$$R(AP' - A'P) = K = \mathrm{const}; \quad K = 0 \quad \Rightarrow \quad P = cA, \quad c = \mathrm{const}.$$

Eq. (3) then takes the form

$$[A^3(R/A)']' = 2A.$$

It is a single equation for two functions $A(x)$ and $R(x)$. It is solved by quadratures if one specifies $A(x)$: indeed, we then obtain

$$\left(\frac{R}{A}\right)' = \frac{2}{A^3}\int A(x)dx. \qquad (8)$$

A case of interest for us is that $A \to 1$ as $x \to -\infty$ (flat space $\times\mathbb{T}^2$) and $A \sim x^2$ as $x \to +\infty$ (AdS$_6$).

It is hard to find such $A(x)$ leading to good analytic expressions of other quantities. We therefore choose a piecewise smooth function $A(x)$:

$$A(x) = \begin{cases} 1, & x \le 0, \\ 1 + 3x^2/a^2, & x \ge 0, \end{cases} \qquad a = \mathrm{const} > 0,$$

solve the equations separately for $x < 0$ and $x > 0$ and match the solutions at $x = 0$. At $x < 0$ we have $R'' = 2$, hence we can take

$$R(x) \equiv r^2(x) = x^2 + b^2, \qquad b = \mathrm{const} > 0 \qquad (x \le 0),$$

thus $x = 0$ is a throat of radius b. Also, without loss of generality,

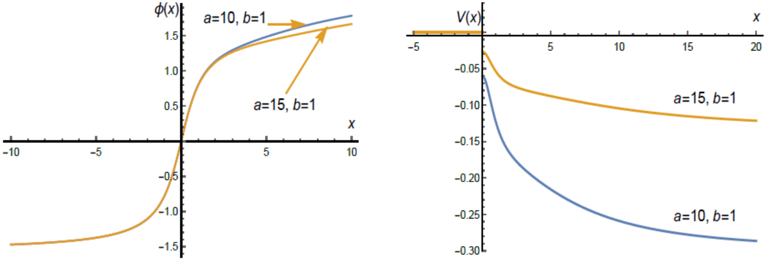
$$V(x) \equiv 0, \qquad \phi(x) = \arctan(x/b) \qquad (x \le 0).$$

At $x > 0$ we obtain

$$R(x) = \left(1 + \frac{3x^2}{a^2}\right)\left[b^2 + \frac{x^2(1 + 2x^2/a^2)}{(1 + 3x^2/a^2)^2}\right],$$

$$V(x) = -\frac{30}{a^2} + \frac{12[b^2x^2 + a^2(2b^2 + x^2)]}{9b^2x^4 + a^4(b^2 + x^2) + 2a^2x^2(3b^2 + x^2)}.$$

Example 2: continued 2



The scalar field $\phi(x)$ (left) and the potential $V(x)$ (right) in Example 2.

- $\phi'(x)$ and $V(x)$ have jumps at $x = 0$, easily smoothed by small changes in specifying $A(x)$.
- Since $P(x) = cA(x)$ (c arbitrary), choosing c, we can make the extra dimensions arbitrarily small on the left end;
- on the right end we have 6D AdS;
- the throat radius b is also arbitrary.

Conclusion

- In 6D GR, we have found examples of wormholes which lead from our universe with small extra dimensions to a universe with large extra dimensions where space-time is effectively 6-dimensional and should contain quite unusual physics.
- In our explicit examples the extra dimensions have the geometry of a 2-torus. Other geometries, topologies and numbers of dimensions are possible and are of interest.
- Other opportunities in the same framework can also be implemented, such as, for example, a de Sitter asymptotic leading to space-times with horizons and very probably to new cosmological models of "black universe" type, where the cosmological expansion starts from a Killing horizon instead of a singularity [KB and J. Fabris, 2006; S. Bolokhov, KB and MS, 2012, etc.)
- One more subject of a future study can be similar configurations in multidimensional gravity with curvature-nonlinear actions [KB and S. Rubin, 2005–2012; S. Rubin, 2016].
- Of utmost interest are possible observational properties of this and other kinds of multidimensional models of gravity.

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THANK YOU!