Wormholes leading to extra dimensions

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The 2nd International Conference
on Particle Physics and Astrophysics
10-14 October 2016, Milan Hotel, Moscow
• Multidimensional theories: a great variety of geometries, topologies and compactification schemes. Extra dimensions can become large and observable in some part of space $\Rightarrow$ another physics.

• A way to study such possible situations: a search for the corresponding solutions of multidimensional Einstein equations.

• We consider 6D manifolds $M_0 \times M_1 \times M_2$; $M_0$: 2D Lorentzian; each of $M_1$, $M_2$: either $S^2$ (2-sphere) or $T^2$ (2-torus). Source of gravity: a minimally coupled (phantom) scalar field.

• We select the possible asymptotic behaviors of the metric functions compatible with the field equations. Their choice is rather narrow.

• Two examples of wormhole solutions with the expected properties. Our “end”: our 4D space $\times T^2$ (small); “far end”: large $T^2$.

• Example 1: a massless scalar field (based on a well-known more general solution). “Far end”: extra $T^2$ has a large constant size.

• Example 2: a nonzero potential $V(\phi)$; “far end”: 6D AdS, all spatial dimensions are infinite.
Basic equations

**Action:** \[ S = \frac{m_6^2}{2} \int \sqrt{|g_6|} \left[ R_6 + 2 \varepsilon_\phi g^{AB} \partial_A \phi \partial_B \phi - 2 V(\phi) \right], \]

where: \( m_6 \) = 6D Planck mass, \( R_6 \) = 6D Ricci scalar, \( g_6 \) = \( \text{det}(g_{AB}) \), \( \varepsilon_\phi = 1 \) for a normal, canonical scalar field, \( \varepsilon_\phi = -1 \) for a phantom one; \( V(\phi) \) = scalar field potential; \( A, B, \ldots = \overline{0, 5} \).

**Equations:** \[ 2 \varepsilon_\phi \Box_6 \phi + \frac{dV}{d\phi} = 0 \]

\[ R^A_B = -\tilde{T}^A_B \equiv -T^A_B - \frac{1}{4} \delta^A_B T^C_C \equiv -2 \varepsilon_\phi \partial^A \phi \partial_B \phi + \frac{1}{2} V(\phi) \delta^A_B, \]

\( R^A_B \) = 6D Ricci tensor, \( T^A_B \) = stress-energy tensor (SET) of \( \phi \).

**Metric:** \[ ds^2 = A(x) dt^2 - \frac{du^2}{A(x)} - R(x) d\Omega_1^2 - P(x) d\Omega_2^2, \]

\( x \) = “radial” coordinate:

\( d\Omega_1^2, d\Omega_2^2 \) = \( x \)-independent metrics on 2D manifolds of unit size;

\( R(x) = r^2(x) \) = size of \( \mathbb{M}_1 \) (2-sphere or 2-torus);

\( P(x) = p^2(x) \) = size of \( \mathbb{M}_2 \) (2-sphere or 2-torus);

**Scalar field:** \( \phi = \phi(x) \).
Which of $\mathbb{M}_{1,2}$ belongs to our 4D space-time and which is “extra”? Everything depends on their size.

**SET:**

\[
\begin{align*}
\tilde{T}_t^t &= \tilde{T}_2^2 = \tilde{T}_3^3 = \tilde{T}_4^4 = \tilde{T}_5^5 = -\frac{1}{2} V(\phi), \\
\tilde{T}_t^t - \tilde{T}_u^u &= 2\varepsilon_\phi A(x)\phi'^2.
\end{align*}
\]

Symmetry of the problem $\Rightarrow$ 4 independent equations (prime $= d/dx$):

\[
\begin{align*}
R^t_t - \tilde{R}^2_x &= 0 & \Rightarrow \quad & [P(AR' - A'R)]' = 2\varepsilon_1 P, \\
R^t_t - R^4_x &= 0 & \Rightarrow \quad & [R(AP' - A'P)]' = 2\varepsilon_2 R.
\end{align*}
\]

($\varepsilon_1 = 1 \iff \mathbb{M}_1 = \text{sphere}, \quad \varepsilon_1 = 0 \iff \mathbb{M}_1 = \text{torus}$. The same for $\varepsilon_2$.)

Note: In Eqs. (3) and (4) — only metric functions! 2 eqs for 3 unknowns.

If we know $A(x), R(x), P(x)$, we find $V(x)$ and $\phi(x)$ from (1) and (2).
Eq. (2) ⇒ solutions with $r > 0$ and $p > 0$ in the whole range $x \in \mathbb{R}$ exist only with $\varepsilon_\phi = -1$, i.e., a phantom field, since they require $r'' > 0$ and $p'' > 0$.

- **SS (double spherical) space-times:** $\varepsilon_1 = \varepsilon_2 = 1$. If spheres $M_1$ are large and $M_2$ are small (or vice versa), there is static spherical symmetry in our space-time and a spherical extra space. Both spheres are large ⇒ 6D space-time, all dimensions are observable.

- **ST (spherical-toroidal) space-times:** the case $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ (or vice versa). If $M_1$ is large and $M_2$ small, we have static spherical symmetry in our space-time and a toroidal extra space. The opposite situation is also possible as well as a total observable 6D geometry.

- **TT (double toroidal) space-times:** if $\varepsilon_1 = \varepsilon_2 = 0$, we have the same as before but both $M_1$ and $M_2$ are toroidal.

**Our interest:** finding configurations where $x \in \mathbb{R}$ and there are different asymptotic behaviors of $R = r^2$ and $P = p^2$ as $x \to \pm \infty$. 
We check which kinds of asymptotics (4D or 6D flat, dS, AdS) are admitted by the pure metric equations (3) and (4):

\[ [P(AR' - A'R)]' = 2\varepsilon_1 P \quad (3), \quad [R(AP' - A'P)]' = 2\varepsilon_2 R. \quad (4) \]

Example of asymptotic analysis. Consider an asymptotically flat 4D spherically symmetric space-time with constant spherical extra dimensions. This means \( \varepsilon_1 = \varepsilon_2 = 1 \) and, without loss of generality (\( \text{fin} \equiv \text{const} > 0 \)),

\[ A(x) \to \text{fin}, \quad R(x) \sim x^2, \quad P(x) \to \text{fin} \quad (5) \]

as \( x \to \infty \). We substitute to (3) and (4) the expansions

\[ A(x) = A_0 + \frac{A_1}{x} + \ldots, \quad R(x) = x^2(1 + o(1)), \quad P(x) = P_0 + \frac{P_1}{x} + \ldots, \]

so that \( R' \sim x, \quad A' \sim x^{-2} \) or even smaller, and the l.h.s. of (3) tends, in general, to a nonzero constant, which agrees with \( P \to \text{fin} \) on the r.h.s. However, in (4) the expression in square brackets tends to a constant, hence its derivative vanishes, while the r.h.s., equal to \( 2R \), should behave as \( x^2 \). Thus the conditions (5) are incompatible with the field equations. The same follows if consider \( x \to -\infty \) and/or exchange \( R(x) \) and \( P(x) \).
In the above manner we analyze different opportunities and obtain the table:

<table>
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<th>6D geometries</th>
<th>Comments</th>
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Here: + (plus) means “possible”, − (minus) — “impossible”, ± — possible under special conditions on the parameters. $M$ stands for Minkowski; the comment “dS” means a de Sitter asymptotic with $A(x) \sim -x^2$, whereas an AdS behavior ($A \sim x^2$) is impossible.

In particular, wormholes with $M^4 \times T^2$ on one or both ends are only possible with ST geometry. Further on: two examples of such wormholes.
Example 1: ST wormholes with a massless scalar

It is a special case of multidimensional solutions with a massless scalar known for a long time (KB, 1995, KB, Ivashchuk and Melnikov, 1997, etc.)

**Metric and scalar field:**

\[ ds^2 = dt^2 - e^{-4nu} \left[ dz^2 + (z^2 + \bar{k}^2) d\Omega_1^2 \right] - e^{2nu} d\Omega_2^2, \]

\[ \phi = Cu, \quad u := \frac{1}{k} \cot^{-1} \left( \frac{-z}{\bar{k}} \right), \]  \hspace{2cm} (6)

Here, \( \bar{k}, \ n, \ C \) = integration constants, such that \( \bar{k}^2 + 3n^2 = 2C^2 \).

It is a spherically symmetric, twice asymptotically flat wormhole in 4D subspace \( \mathbb{M}_0 \times \mathbb{M}_1 (M_1 = \mathbb{S}^2) \) with a toroidal extra space \( \mathbb{M}_2 = \mathbb{T}^2 \).

**Note:** \( z \) is another coordinate than \( x \) used in other parts of this presentation.
Size of $\mathbb{T}^2$: $p = p_− = 1$ ($z = −∞$) — “here”,
$p = p_+ = e^{n\pi/\kappa} p_−$ ($z = +∞$) — at the “far end”.

**Wormhole throat:** a minimum of $r(z) = e^{-2nu(z^2 + \kappa^2)^{1/2}}$, located at $z = 2n$, its radius:

$$r_{\text{min}} = \sqrt{\kappa^2 + 4n^2 \exp \left(\frac{2n}{\kappa} \cot^{-1} \frac{2n}{\kappa}\right)}.$$  \hspace{1cm} (7)

Suppose that the size of extra dimensions $p_−$ on “our” end, $z = −∞$, is small enough to be invisible by modern instruments, say,

$$p_− = 10^{-17} \text{ cm}.$$

On the other end, it depends on the ratio $n/\kappa$.
Thus, to obtain $p = p_+ \sim 1 \text{ m}$, we should take $n/\kappa \approx 14$.

The **throat radius** also depends on $n$ and $\kappa$. It is not too large if they take modest values. Thus, for $n/\kappa = 14$, we have $r_{\text{min}} \approx 76\kappa p_−$. To obtain a large enough throat for passing of a macroscopic body, say, $r_{\text{min}} = 10 \text{ meters}$, one has to suppose $\kappa \sim 10^{18}$.  

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Example 2: ST asymptotically AdS wormholes

With nonzero potentials $V(\phi)$, in most cases solutions can be found only numerically, with one exception (recall that $\varepsilon_2 = 0$ in the ST case!):

$$R(AP' - A'P) = K = \text{const}; \quad K = 0 \Rightarrow P = cA, \quad c = \text{const}.$$  

Eq. (3) then takes the form

$$[A^3(R/A)']' = 2A.$$  

It is a single equation for two functions $A(x)$ and $R(x)$. It is solved by quadratures if one specifies $A(x)$: indeed, we then obtain

$$\left(\frac{R}{A}\right)' = \frac{2}{A^3} \int A(x)dx. \quad (8)$$

A case of interest for us is that $A \rightarrow 1$ as $x \rightarrow -\infty$ (flat space $\times \mathbb{T}^2$) and $A \sim x^2$ as $x \rightarrow +\infty$ (AdS$_6$).
Example 2: continued

It is hard to find such $A(x)$ leading to good analytic expressions of other quantities. We therefore choose a piecewise smooth function $A(x)$:

$$A(x) = \begin{cases} 
1, & x \leq 0, \\
1 + 3x^2/a^2, & x \geq 0,
\end{cases} \quad a = \text{const} > 0,$$

solve the equations separately for $x < 0$ and $x > 0$ and match the solutions at $x = 0$. At $x < 0$ we have $R'' = 2$, hence we can take

$$R(x) \equiv r^2(x) = x^2 + b^2, \quad b = \text{const} > 0 \quad (x \leq 0),$$

thus $x = 0$ is a throat of radius $b$. Also, without loss of generality,

$$V(x) \equiv 0, \quad \phi(x) = \arctan(x/b) \quad (x \leq 0).$$

At $x > 0$ we obtain

$$R(x) = \left(1 + \frac{3x^2}{a^2}\right)\left[b^2 + \frac{x^2(1 + 2x^2/a^2)}{(1 + 3x^2/a^2)^2}\right],$$

$$V(x) = -\frac{30}{a^2} + \frac{12[b^2x^2 + a^2(2b^2 + x^2)]}{9b^2x^4 + a^4(b^2 + x^2) + 2a^2x^2(3b^2 + x^2)}.$$
Example 2: continued 2

The scalar field $\phi(x)$ (left) and the potential $V(x)$ (right) in Example 2.

- $\phi'(x)$ and $V(x)$ have jumps at $x = 0$, easily smoothed by small changes in specifying $A(x)$.

- Since $P(x) = cA(x)$ ($c$ arbitrary), choosing $c$, we can make the extra dimensions arbitrarily small on the left end;

- on the right end we have 6D AdS;

- the throat radius $b$ is also arbitrary.
In 6D GR, we have found examples of wormholes which lead from our universe with small extra dimensions to a universe with large extra dimensions where space-time is effectively 6-dimensional and should contain quite unusual physics.

In our explicit examples the extra dimensions have the geometry of a 2-torus. Other geometries, topologies and numbers of dimensions are possible and are of interest.

Other opportunities in the same framework can also be implemented, such as, for example, a de Sitter asymptotic leading to space-times with horizons and very probably to new cosmological models of “black universe” type, where the cosmological expansion starts from a Killing horizon instead of a singularity [KB and J. Fabris, 2006; S. Bolokhov, KB and MS, 2012, etc.)

One more subject of a future study can be similar configurations in multidimensional gravity with curvature-nonlinear actions [KB and S. Rubin, 2005–2012; S. Rubin, 2016].

Of utmost interest are possible observational properties of this and other kinds of multidimensional models of gravity.
