Relic gravitational waves in cosmological models based on Einstein-Gauss-Bonnet gravity

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General Relativity. Cosmological solutions

Considered action ($8\pi G = c = 1$ units system)

$$S_E = \int d^4x \left[\frac{1}{2}R - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right]. \qquad (1)$$

$$3H\dot{\phi} + \ddot{\phi} + V'_\phi(\phi) = 0, \quad (2)$$

$$3H^2 = \frac{1}{2}\dot{\phi} + V(\phi), \quad (3)$$

$$2\dot{H} + 3H^2 = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (4)$$

$$V(\phi(t)) = 3H^2 + \dot{H}, \quad (5)$$

$$\dot{\phi}^2 = -2\dot{H}. \quad (6)$$

$$\dot{\phi} = \sqrt{-2\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2} = \sqrt{(f(t))^2}, \quad (7)$$

$$\phi(t) = \int f(t)dt, \quad (8)$$

$$a(t) = a_0 e^{c_1 t} \exp\left[-\frac{1}{2}\int\left(\int f^2(t)dt\right)dt \right], \quad (9)$$

$$H(t) = \int -\frac{1}{2} f^2(t)dt. \quad (10)$$

Background dynamic equations [1]

Analytical solutions generated by function $f(t)$

[1] Chervon S. et al. Scalar field cosmology. – World Scientific, 2019. – T. 13.

Einstein – Gauss – Bonnet gravity

Arises in the low-energy effective action for the heterotic strings, also appears in the second order of Lovelock gravity theory.

Considered action ($8\pi G = c = 1$ units system)

$$S_{GB}=\int d^4x\left[\frac{1}{2}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{GB}\partial_\nu\phi_{GB}-V_{GB}(\phi_{GB})+\xi(\phi_{GB})R_{GB}\right],\quad (11)$$

where $R_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, ξ – is the nonminimal coupling function [2]

$$3H_{GB}^2=\frac{1}{2}\dot{\phi}_{GB}^2+V_{GB}(\phi_{GB})+12\dot{\xi}H_{GB}^3,\quad (11)$$

$$\dot{\phi}_{GB}^2=-2\dot{H}_{GB}+4\ddot{\xi}H_{GB}^2+4\dot{\xi}H_{GB}\left(2\dot{H}_{GB}-H_{GB}^2\right),\quad (12)$$

$$\ddot{\phi}_{GB}+3H_{GB}\dot{\phi}_{GB}+\partial_{\phi_{GB}}V_{GB}(\phi_{GB})+12H_{GB}^2\left(\dot{H}_{GB}+H_{GB}^2\right)\partial_{\phi_{GB}}\xi\left(\phi_{GB}\right)=0,\quad (13)$$

$$V_{GB}\left(\phi_{GB}\right)=3H_{GB}^2+\dot{H}_{GB}-10H_{GB}^3\dot{\xi}-2H_{GB}^2\ddot{\xi}_{GB}-4H_{GB}\dot{H}_{GB}\dot{\xi},\quad (14)$$

$$\frac{1}{2}\dot{\phi}_{GB}=-\dot{H}_{GB}-H_{GB}^3\dot{\xi}+4H_{GB}\dot{H}_{GB}\dot{\xi}+2H_{GB}^2\ddot{\xi},\quad (15)$$

Taking into account connection $H_E = H_{GB}\left(1 - 2\dot{\xi}H_{GB}\right)$ [2] and by introducing deviation parameter Δ_H

$$\Delta_H=H_{GB}-H_E,\quad (16)$$

$$\Delta_H=-\beta\left(\frac{\dot{H}_E}{H_E}+\frac{\ddot{H}_E}{2\dot{H}_E}\right),\quad \beta=const\quad (17)$$

one can generate analytical solutions for GR and extend them for EGB gravity!

[2] Fomin I. Gauss–Bonnet term corrections in scalar field cosmology//The European Physical Journal C. – 2020. – T. 80. – №. 12. – C. 1145.

Slow-roll parameters and connected solutions

$$\epsilon_E = -\frac{\dot{H}_E}{H_E^2}, \quad (18)$$

$$\delta_E = -\frac{\ddot{H}_E}{2H_E\dot{H}_E}, \quad (19)$$

Slow-roll approximation: $\epsilon \ll 1, \delta \ll 1$

Taking into account
- eq. (5)-(6)
- Deviation parameter eq. (17)
- eq. (18)-(19)
- and that $H_E = H_{GB}\left(1 - 2\dot{\xi}H_{GB}\right)$ [2]

One can redefine deviation parameter Δ_H as

$$\Delta_E = \frac{\Delta_H}{H_E} = \beta\left(\epsilon_E + \delta_E\right), \quad (20)$$

and write EGB solutions via GR cosmological parameters!

$$V_{GB}\left(\phi_{GB}\right) = H_E^2\left(3 - \epsilon_E + \Delta_E - 2\Delta_E^2\right) = 3H_E^2\left(1 + \Delta_E^2\right) - \frac{1}{2}\dot{\phi}_{GB}^2, (21)$$

$$\frac{1}{2}\dot{\phi}_{GB}^2 = H_E^2\left(\epsilon_E - \Delta_E - \Delta_E^2\right), (22)$$

$$\dot{\xi} = \frac{1}{H_E}\frac{\Delta_E}{\left(1+\Delta_E\right)^2}, (23)$$

$$\epsilon_{GB} = \frac{\epsilon_E}{1+\Delta_E} - \frac{\dot{\Delta}_E}{H_E\left(1+\Delta_E\right)^2}, (24)$$

$$\delta_{GB} = -\frac{-2\epsilon_E^2 H_E^3\left(1+\Delta_E\right) + H_E^2\left(1+\Delta_E\right)\dot{\epsilon}_E + 3\epsilon_E H_E^2\dot{\Delta}_E + \dot{H}_E\dot{\Delta}_E - H_E\ddot{\Delta}_E}{2H_E^2\left(1+\Delta_E\right)\left(\epsilon_E H_E\left(1+\Delta_E\right) - \dot{\Delta}_E\right)}, \quad (25)$$

$$n_{S(GB)} - 1 = -2\epsilon_{GB} - \frac{1}{H_E\left(1+\Delta_E\right)}\frac{d}{dt}\ln\left[\epsilon_{GB} - \frac{\Delta_E}{1+\Delta_E}\right], (26)$$

$$r_{GB} = 16\left(\epsilon_{GB} - \frac{\Delta_E}{1+\Delta_E}\right). (27)$$

Experimental constrains [3,4]

$$n_S = 0.965 \pm 0.004,$$

$$r < 0.032,$$

[3] Aghanim N., et al. Planck 2018 results-VI. Cosmological parameters // Astronomy & Astrophysics. — 2020. — Vol. 641. — P. A6.
[4] Tristram M., et al. Improved limits on the tensor-to-scalar ratio using BICEP and P l a n c k data // Physical Review D. — 2022. — Vol. 105, no. 8. — P. 083524.

Special class of cosmological solutions

One can consider a special type of cosmological inflation when first and second slow roll parameters (18)-(19) have constant relation, i.e.

$$\frac{\delta_E}{\epsilon_E} = \mathcal{C} = const. \quad (28)$$

This allows us to simplify deviation parameter Δ_E as

$$\Delta_E = \beta\left(1+\mathcal{C}\right)\epsilon_E, \quad (29)$$

and rewrite (21)-(23), (26)-(27)

$$V_{GB}\left(\phi_{GB}\right) = 3H_E^2\left(1+\beta^2\left(1+\mathcal{C}\right)^2\epsilon_E^2\right) - \frac{1}{2}\dot{\phi}_{GB}^2, \quad (30)$$

$$\frac{1}{2}\dot{\phi}_{GB}^2 = H_E^2\left(\epsilon_E - \beta\left(1+\mathcal{C}\right)\epsilon_E\left(1+\beta\left(1+\mathcal{C}\right)\epsilon_E\right)\right), \quad (31)$$

$$\dot{\xi} = \frac{1}{H_E}\frac{\beta\left(1+\mathcal{C}\right)\epsilon_E}{\left(1+\beta\left(1+\mathcal{C}\right)\epsilon_E\right)^2}, \quad (33)$$

$$n_{S(GB)} - 1 = -\frac{2\left((\beta(\mathcal{C}+1)+\mathcal{C}-1)\epsilon_E^2+\epsilon_E\right)}{\left(\beta(\mathcal{C}+1)\epsilon_E+1\right)^2} - \frac{2(1-\mathcal{C})\epsilon_E^2\left(-\left(\beta^2(\mathcal{C}+1)^2+\beta(\mathcal{C}+1)+2(1-\mathcal{C})\right)\epsilon_E-(\beta(\mathcal{C}+1))+1\right)}{\left(\beta(\mathcal{C}+1)\epsilon_E+1\right)^2\left(-\left(\left(\beta^2(\mathcal{C}+1)^2-\beta(\mathcal{C}+1)-\mathcal{C}+1\right)\epsilon_E^2\right)-\beta(\mathcal{C}+1)+1\right)}, \quad (34)$$

$$r_{GB} = \frac{16\left(\epsilon_E^2\left(\beta^2\left(-(\mathcal{C}+1)^2\right)+\beta(\mathcal{C}+1)+\mathcal{C}-1\right)+\epsilon_E(1-\beta(\mathcal{C}+1))\right)}{\left(\beta(\mathcal{C}+1)\epsilon_E+1\right)^2}. \quad (35)$$

Verification by $r(n_S)$ dependency

By setting ϵ_E to be a parameter in equations (34)-(35) one can obtain parametrical dependency of $r(n_S)$ and set constrains for model constants β and $\mathcal{C}$:

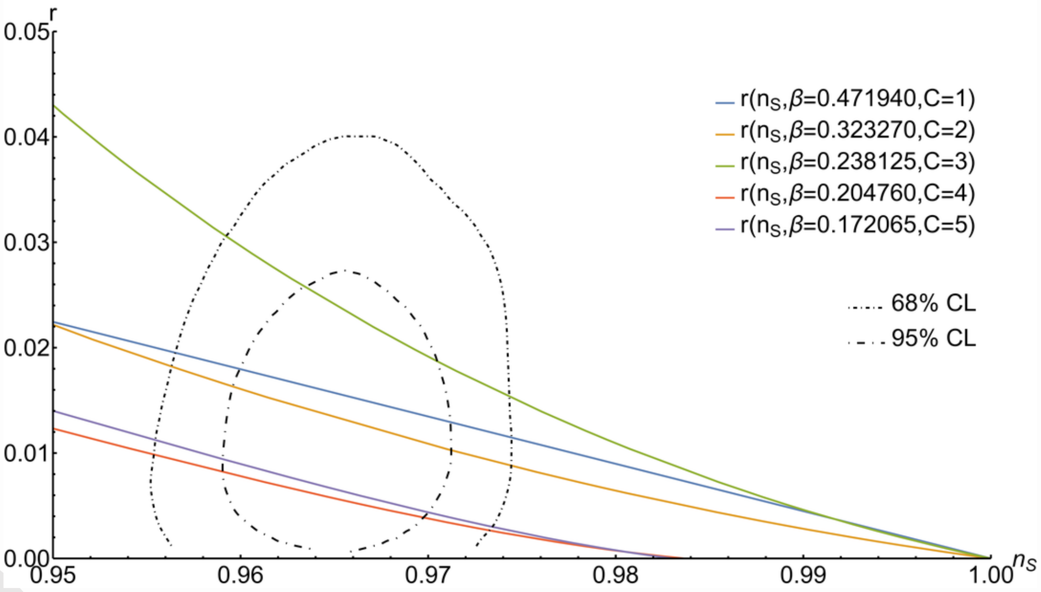

Fig.1. $r(n_S)$ dependency for various values of $\mathcal{C}$ and β

$\mathcal{C}$	β
1	[0.44388, 0.50000)
2	[0.30691, 0.33963]
3	[0.22900, 0.24725]
4	(0.20000, 0.20952)
5	(0.16752, 0.17661)

Table 1. Ranges of β corresponding to the values of $\mathcal{C}$ in which the models can be verified by the dependence of the tensor-scalar relation on the slope of the scalar perturbation spectrum

Speed of tensor and scalar perturbations propagation

Speed of scalar perturbations (speed of light)

$$c_S^2 - 1 = -\frac{2\beta^2(\mathcal{C}+1)^2\epsilon_E^{\,2}\left(\beta^3(\mathcal{C}+1)^3\epsilon_E^{\,2}((\mathcal{C}-1)\epsilon_E+1)+\beta^2(\mathcal{C}+1)^2\epsilon_E((2\mathcal{C}-9)\epsilon_E+2)-\beta(\mathcal{C}+1)(\epsilon_E(7(\mathcal{C}-1)\epsilon_E-\mathcal{C}+10)-1)-2(\mathcal{C}-1)\epsilon_E-2\right)}{(\beta(\mathcal{C}+1)\epsilon_E-1)\left(\beta^4(\mathcal{C}+1)^4\epsilon_E^{\,3}((\mathcal{C}-1)\epsilon_E+4)+\beta^3(\mathcal{C}+1)^3\epsilon_E^{\,2}((\mathcal{C}-1)\epsilon_E+7)-\beta^2(\mathcal{C}+1)^2\epsilon_E(\mathcal{C}\epsilon_E-2)-\beta(\mathcal{C}+1)((\mathcal{C}-1)\epsilon_E(\epsilon_E+1)+1)+(\mathcal{C}-1)\epsilon_E+1\right)}, \quad (36)$$

Speed of tensor perturbations (speed of gravitational waves)

$$c_G^2 - 1 = \frac{2\beta\left(1+\mathcal{C}\right)\epsilon_E\left(1+\left(-1+\mathcal{C}\right)\epsilon_E-\frac{\epsilon_E\left(1+\left(-1+\beta+\mathcal{C}+\beta\mathcal{C}\right)\epsilon_E\right)}{\left(1+\beta\left(1+\mathcal{C}\right)\epsilon_E\right)^2}\right)}{1-\beta\left(1+\mathcal{C}\right)\epsilon_E}, \quad (37)$$

When $\beta \to 0$ and (or) $\epsilon_E \to 0$ $c_S = c_G = 1$ (in chosen system of units)!
This is actually good, taking into account GW170817 event [5]!

[5] Abbott B. P. Multi-messenger observations of a binary neutron star merger. – 2017.

Spectrum and detection possibility of relic GW

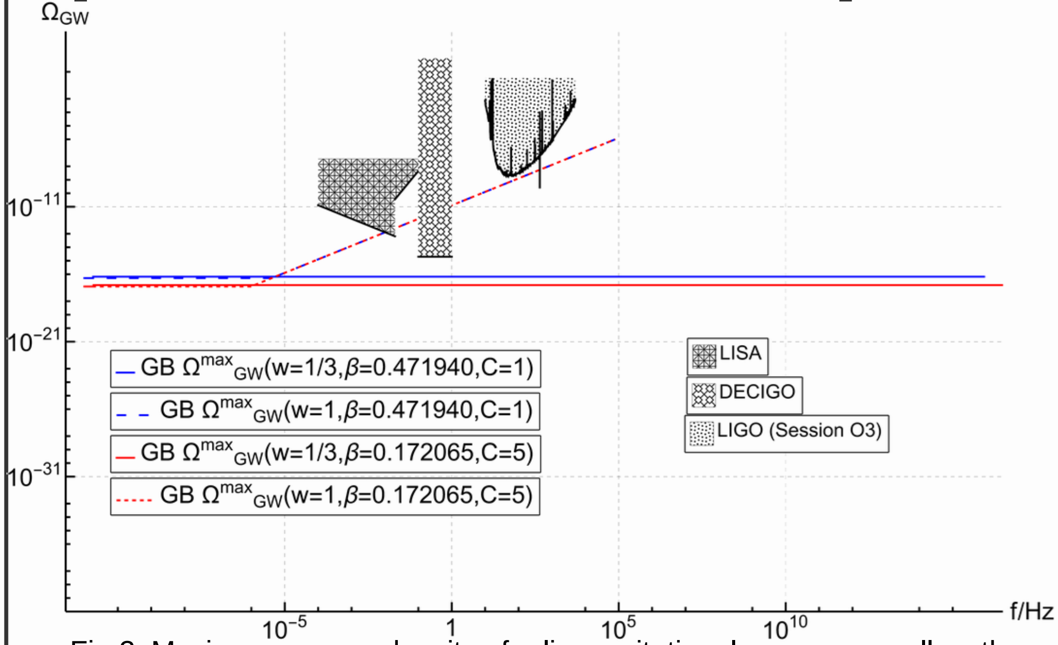

Fig.2. Maximum energy density of relic gravitational waves, as well as the detection bands of LIGO, DECIGO, LISA

Parameter of state

$$w=\boxed{-1+\frac{2}{3}\epsilon_E}\cdot\frac{\beta(\mathcal{C}+1)\left(\beta(\mathcal{C}+1)\epsilon_E+1\right)-1}{\beta^2(\mathcal{C}+1)^2\epsilon_E^2-1},\quad (38)$$

Dimensionless energy density of relic GW [6]

$$\Omega_{gw}\simeq\Omega_{gw}^0\cdot\begin{cases}1, & f<f_{RD}\\ 1.27\left(\dfrac{f}{f_{RD}}\right)^{\alpha_S} & f\geq f_{RD}\end{cases},\quad (39)$$

The cutoff frequency of relic GW [7]

$$f_{cutoff}\leq f_{RD}\left(\alpha_S\left[\frac{1\times10^{-6}}{1.27\times\Omega_{gw}^0}-\frac{1}{1.27}\ln\left(\frac{f_{RD}}{f_{bbn}}\right)\right]+1\right)^{1/\alpha_S},(40)$$

Another possible scheme of verification is low-frequency gravitational-optical resonance [8]

[6] Tanin E. H., Tankanen T., Gravitational wave constraints on the observable inflation. J. Cosm. Astropart. Phys., 2021, vol. 2021, no. 01, p. 053.
[7] Manucharyan G. D., Fomin I. V., Gladyshev V. O., Litvinov D. A. On the detection of high-frequency relic gravitational waves. Space, Time and Fundamental Interactions, 2023, no. 3-4, pp. 188–198.
[8] Фомин И. В., Червон С. В., Морозов А. Н. Гравитационные волны ранней Вселенной //М.: Изд-во МГТУ им. НЭ Баумана. – 2018.

Conclusion

- Presented approach allows generating analytical solutions for GR and thus finding analytical solutions for EGB gravity
- These type of solutions can be verified by tensor-scalar ratio dependency on scalar tilt
- The speed of scalar and tensor perturbation propagation in these type of models fit to current experimental data
- Detection of relic gravitational waves in these type of models is possible on advanced GW detectors like LISA or DECIGO or via low-frequency gravitational-optical resonance

Thanks!

Do you have any questions?

Appendix

- $\Omega^0_{GW} = 10^{-15} \cdot r/h^2$,
- $h \simeq 0.68$ is relative Hubble constant,
- parameter α_S is defined as $\alpha_S = 2\left(\frac{3w-1}{3w+1}\right)$,
- f_{RD} is the frequency of the mode that corresponds to the size of the horizon at the beginning of the epoch of radiation domination, in the present epoch, determined through the LIGO sensitivity constraint $h^2\Omega_{GW}(100) \lesssim 10^{-9}$ as

$$f_{RD}(w) = 100 \cdot \left(\frac{h^2 \cdot \Omega_{GW}(100)}{10^{-15} \cdot r}\right)$$

- The cutoff frequency of the of the spectrum of the relic gravitational-wave background is estimated taking into account the integral condition on the energy density

$$\int_{f_{bbn}}^{f_{cutoff}} \Omega_{GW}(f)\frac{df}{f} \lesssim 1 \times 10^{-6}$$

where $f_{bbn} \simeq 1.8 \times 10^{-11}$Hz is the frequency of the mode corresponding to the size of the horizon at the beginning of the Big Bang Nucleosynthesis