# RECONSTRUCTION THE SCALAR-TORSION GRAVITY VERSION FROM THE FRAME OF EXACT COSMOLOGICAL SOLUTIONS

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# CONTENTS

# INTRODUCTION

Scalar field inflationary models are actively used to describe both stages of accelerated the universe expansion.

Ways to construct and analyze

possible physical effects analysis induced by the different versions of the gravity theories [1]

model's parameters reconstruction from the various observational constraints [2]

Scalar-torsion gravity $f(T,\varphi)$ is widely applied to construct verified cosmological models based on non-minimal coupling of torsion and a scalar field.

[1] D. Baumann, L. McAllister, Inflation and String Theory, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2015).
[2] S. D. Odintsov and V. K. Oikonomou, Reconstruction of Slow-roll F(R) Gravity Inflation from the Observational Indices, Annals Phys. 388, 267-275 (2018).

# GENERALIZED SCALAR-TORSION GRAVITY (GSTG)

Units system: $c = 8\pi G = M_p^{-2} = 1.$

$$S_{(GSTG)} = \int d^4x\sqrt{-g}F(T,X,\varphi) = \int d^4x\sqrt{-g}\left[f(T,\varphi) + \frac{\omega(\varphi)}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right] \text{ [3]}, \quad (1)$$

$g$ – the space-time metric determinant of metric tensor $g_{\mu\nu} = e^A{}_\mu\, e^B{}_\nu \eta_{AB}$, $e = \det\left(e^A{}_\mu\right) = \sqrt{-g}$,

$\eta_{AB} = diag(-1,1,1,1)$ –the Minkowski tangent space metric,

$\omega(\varphi)$- a kinetic function, $T = 6H^2$- a torsion scalar.

$f(T,\varphi)$- is an arbitrary differentiable function

$e^A{}_\mu = diag(1,a,a,a)$ – a proper tetrad is chosen so that $\omega^A{}_{B\mu} = 0.$

## Friedmann-Robertson-Walker (FRW) metric

A homogenous and isotropic universe is described by spatially-flat metric
$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j,$

$a(t)$ – is a scale factor, $t$ – cosmic time.

[3] M. Gonzalez-Espinoza and G. Otalora, Cosmological dynamics of dark energy in scalartorsion f(T, ϕ) gravity, Eur. Phys. J. C 81, no.5, 480 (2021).

# COSMOLOGICAL BACKGROUND EQUATIONS

Tetrad variation of action (1) leads to background equations

$$f(T,\varphi) - \frac{\omega(\varphi)}{2}\dot{\varphi}^2 - 2Tf_{,T} = 0, \qquad (2)$$

$$f(T,\phi) + \frac{\omega(\varphi)}{2}\dot{\varphi}^2 - 2Tf_{,T} - 4\frac{d}{dt}\left(H, f_{,T}\right) = 0, \qquad (3)$$

$$\omega(\varphi)\ddot{\varphi} + 3\omega(\varphi)H\dot{\varphi} + \frac{1}{2}\frac{d\omega(\varphi)}{d\varphi}\dot{\varphi}^2 - f_{,\varphi} = 0 \qquad (4)$$

## Exact solutions

$$f(T,\varphi) = \alpha_1 G(\varphi)\sqrt{T} + \alpha_2 V(\phi) = f(T,\varphi)_{STG} + \alpha_2 V(\varphi), \qquad (5)$$

$$\omega(\varphi) = \frac{\alpha_1^2}{3\alpha_2}\frac{G_{,\varphi}^2}{V(\varphi)}, \qquad (6)$$

$$\dot{\varphi} = \frac{\sqrt{6}\alpha_2}{\alpha_1}\frac{V(\varphi)}{G_{,\varphi}} \qquad (7)$$

$G(\varphi)$- an arbitrary differentiable function, where $\dot{G} = \frac{\sqrt{6}\alpha_2}{\alpha_1}V(\varphi)$.

# COSMOLOGICAL PERTURBATIONS

$$\mathcal{P}_S = \frac{H^2}{8\pi^2 Q_s}\left[1 + \eta_{\mathcal{R}} \ln\left(\frac{k}{aH}\right)\right]\Big|_{k=aH} = \frac{H^2}{8\pi^2 Q_s}$$ – power spectrum of scalar perturbations [4]

$$\mathcal{P}_T = \frac{H^2}{2\pi^2 Q_T}$$ – power spectrum of tensor perturbations

$$Q_s = \frac{\omega(\phi)X}{H^2}, Q_T = -\frac{1}{2} f_{,T}$$ – the parameters.

$$n_s - 1 = -2\epsilon - \eta + 2\eta_R$$ – spectral index of scalar perturbations

$$r = \frac{P_T}{P_S} \simeq 16\delta_{\omega X}$$ –tensor-to-scalar ratio.

## Slow-roll approximation parameters

$$\epsilon = -\frac{\dot{H}}{H^2}, \delta_{\omega X} = -\frac{\omega X}{2H^2 f_{,T}}, \delta_{f,T} = \frac{\dot{f}_{,T}}{H\, f_{,T}}, \delta_{f\dot{H}} = \frac{f_{,TT}\dot{T}}{H\, f_{,T}},$$

$$\delta_{fX} = \frac{f_{,T\varphi}\dot{\varphi}}{H f_{,T}}, \eta_R = \delta_{f,T}\left[1 + \frac{\delta_{f,T}}{\delta_{f\dot{H}}}\left(1 + \frac{\delta_{fX}}{\delta_{\omega X}}\right)\right],$$

$$\eta = \frac{\dot{Q}_S}{HQ_S} = 3\epsilon + \frac{1}{H}\left[\frac{d}{dt}\ln(\omega X H)\right] = 2\epsilon + \frac{1}{H}\left(\frac{\dot{\omega}}{\omega} + \frac{\dot{X}}{X}\right).$$

[4] M. Gonzalez-Espinoza and G. Otalora, Generating primordial fluctuations from modified teleparallel gravity with local Lorentz-symmetry breaking, Phys. Lett. B 809, 135696 (2020)

# Slow-roll approximation

On the basis of exact solutions $(5)-(7)$ we obtain

$$\omega X = \frac{\alpha_1}{\sqrt{6}}\dot{G} = \alpha_2 V.$$

Thus, slow-roll parameters are

$$\delta_{\omega X} = -\frac{\alpha_1 \dot{G}}{HG}, \delta_{f,T} = \epsilon + \delta_{fX}, \delta_{f\dot{H}} = \epsilon,$$

$$\delta_{fX} = \frac{\sqrt{6}\alpha_2}{\alpha_1}\frac{V}{HG}, = \frac{\dot{G}}{HG} = \frac{\delta_{\omega X}}{\alpha_1}, \eta = 2\epsilon + \frac{\ddot{G}}{H\dot{G}},$$

$$\eta_R = \delta_{f,T}\left[1 + \left(1 + \frac{\delta_{fX}}{\epsilon}\right)\left(1 + \frac{1}{\alpha_1}\right)\right] = \left(\epsilon + \frac{\dot{G}}{HG}\right)\left[1 + \left(1 - \frac{H\dot{G}}{G\dot{H}}\right)\left(1 + \frac{1}{\alpha_1}\right)\right].$$

# COSMOLOGICAL MODELS VERIFICATION

Modern restrictions on cosmological perturbation parameters [5]

$$\mathcal{P}_s = 2.1 \cdot 10^{-9},$$

$$n_s = 0.9663 \pm 0.004,$$

$$r < 0.032.$$

Let us use exact solutions (5)-(7) ant rewrite this parameters as

$$n_S - 1 = 4\frac{\dot{H}}{H^2} - \frac{\ddot{G}}{H\dot{G}} + 2\left(-\frac{\dot{H}}{H^2} + \frac{\dot{G}}{HG}\right)\left[1 + \left(1 - \frac{H\dot{G}}{G\,\dot{H}}\right)\left(1 + \frac{1}{\alpha_1}\right)\right],$$

$$r = 16\alpha_1\frac{\dot{G}}{HG},$$

$$n_T = 3\frac{\dot{H}}{H^2} - \frac{\dot{G}}{HG} = -3\epsilon - \frac{r}{16\alpha_1}.$$

For $\alpha_1 \succ -\frac{r}{48\epsilon}$, $\alpha_1 < -\frac{r}{48\epsilon}$ and $\alpha_1 = -\frac{r}{48\epsilon}$ - red, blue and flat tilt tensor spectrum respectively.

[5] M. Tristram, A. J. Banday, K. M. G´orski, R. Keskitalo, C. R. Lawrence, K. J. Andersen, R. B. Barreiro, J. Borrill, L. P. L. Colombo and H. K. Eriksen, et al. Improved limits on the tensor-to-scalar ratio using BICEP and Planck data, Phys. Rev. D 105 (2022) no.8, 083524.

# RECONSTRUCTION OF NON-MINIMAL COUPLING FUNCTION FROM LINEAR DEPENDENCE $r \sim (1 - n_S)$

For $n_S \simeq 0.97$ and $1 - n_S \ll 1$ we obtain

$$r = \sum_{k=0}^{\infty} \beta_k (1 - n_s)^k = \beta_0 + \beta_1 (1 - n_S) + \beta_2 (1 - n_S)^2 + \cdots, [6]$$

Initial conditions (flat Harrison-Zeldovich spectrum): $r(n_s = 1) = 0 \Rightarrow r(0) = \beta_0 = 0$.

New specific classification of cosmological models by considering $\beta_1 \neq 0, \beta_2 \neq 0$ etc.

The special case $\alpha_1 = -1$ allow to obtain:

$$n_S - 1 = 2\frac{\dot{H}}{H^2} + \frac{2\dot{G}}{HG} - \frac{\ddot{G}}{H\dot{G}},$$

$$r = -16\frac{\dot{G}}{HG}.$$

The dependence if defined as follows: $r \simeq \frac{8}{m}(1 - n_S), m > 7.5, \beta_1 = \frac{8}{m}$.

[6] Fomin I. V. Cosmological models based on scalar-torsion gravity. Space, Time and Fundamental Interactions, 2024, no. 1, pp. 104–109.

# RECONSTRUCTION OF NON-MINIMAL COUPLING FUNCTION FROM LINEAR DEPENDENCE $r \sim (1 - n_S)$

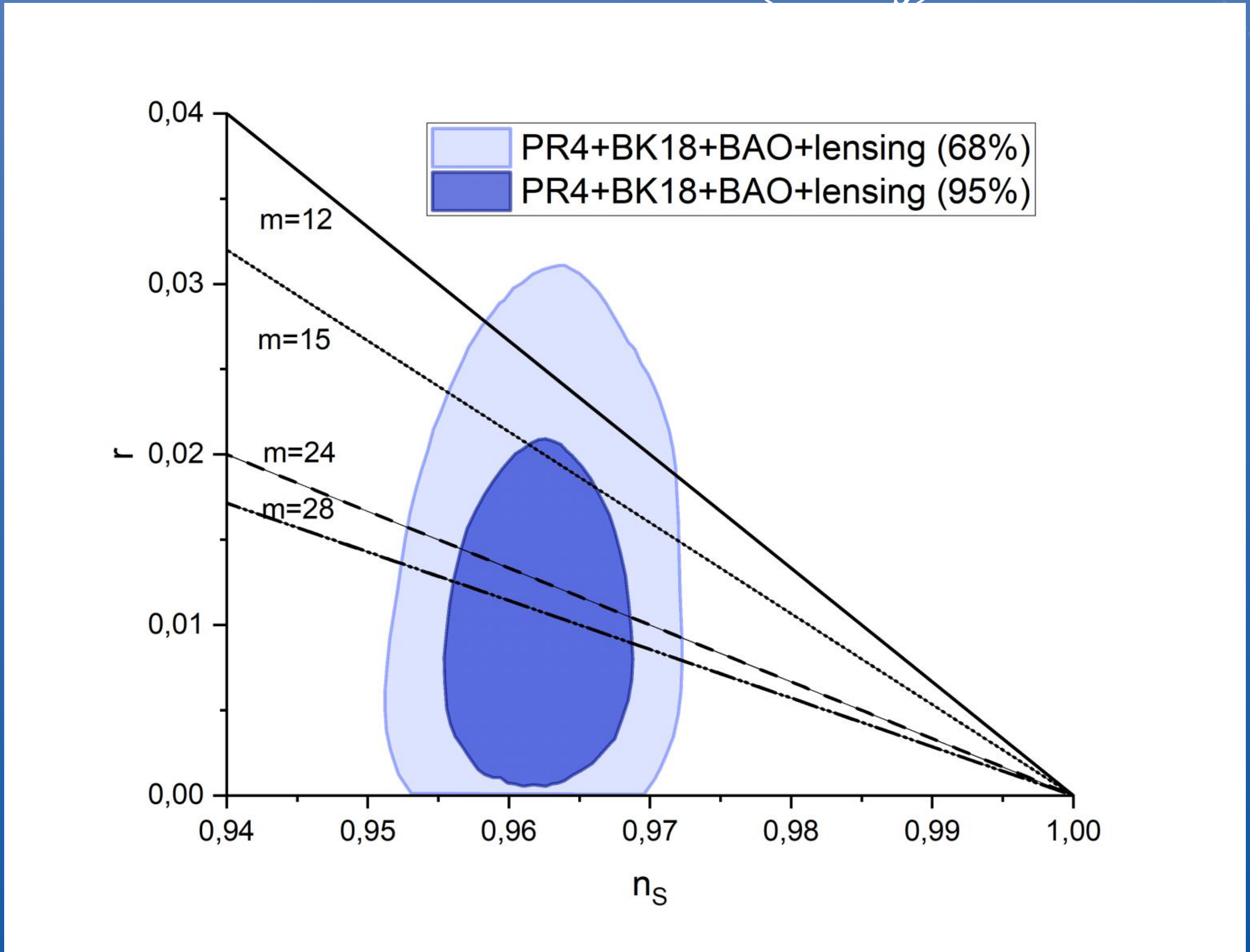


Fig. 1. The dependence $r = r(n_S)$ for different parameter values $m = 12,15,24,28,$, with constraints on the tensor-to-scalar ratio r due to the Planck TT,TE,EE+lowE+lensing+BAO+BICEP2/Keck Array observations.

# COSMOLOGICAL MODELS VERIFICATION

The connection between $H$ and $G$: $H(t) = const \times \frac{\dot{G}^{\frac{1}{2}}}{G^{m-1}} \rightarrow \mathrm{H(t)} = \mathrm{const} \times \frac{V^{\frac{1}{2}}}{G^{m-1}}$.

Then, we obtain $f(T,\varphi)_{STG} = -G(\varphi)\sqrt{T} = \mathrm{const} \times \sqrt{T} \times [V(\varphi)]^{-\frac{k+1}{2(m-1)}}$,

$H(\varphi) = const \times [V(\varphi)]^{-\frac{k}{2}}$.

## Inflation with exponential potential based on $f(T,\varphi)_{STG} = -G(\varphi)\sqrt{T}$

$V(\varphi) = V_0 e^{-2\beta\varphi}$,

$H(\varphi) = \lambda e^{\beta k \varphi}$,

$G(\varphi) = G_0 \exp\left[\frac{\beta(k+1)}{m-1}\varphi\right]$,

$\omega(\varphi) = \omega_0 \exp\left[\frac{2\beta(k+m)}{m-1}\varphi\right]$.

# EXPONENTIAL-POWER LAW INFLATION

$H(t)=\lambda+f(t)$, $f(t)$- a cosmic time function. X [7]

$$\varphi(t)=\frac{1}{\beta k}\ln\left(1+\frac{f(t)}{\lambda}\right).$$

$N(t)=\ln\left(\frac{a}{a_0}\right)=\lambda t+s\ln t$- e-fold number, $a_0-$ the scale factor at the beginning of inflation $(N=0)$.

$f(t)=\frac{s}{t}\Rightarrow H(t)=\lambda+\frac{s}{t}-$ the Hubble parameter,

$a(t)\sim t^{s}e^{\lambda t}-$ the scale factor.

$t(N)$dependence: $t(N)=\exp\left[\frac{N}{s}-W\left(\frac{\lambda}{s}e^{\frac{N}{s}}\right)\right]$, now we obtain $\varphi(N)$ and $V(N)$:

$$\varphi(N)=\frac{1}{\beta k}\ln\left\{1+\frac{s}{\lambda}\exp\left[W\left(\frac{\lambda}{s}e^{\frac{N}{s}}\right)-\frac{N}{s}\right]\right\},$$

$$V(N)=V_0\left\{1+\frac{s}{\lambda}\exp\left[W\left(\frac{\lambda}{s}e^{\frac{N}{s}}\right)-\frac{N}{s}\right]\right\}^{-\frac{2}{k}}.$$

$W-$ the Lambert function.

[7] Fomin, I.; Chervon, S. Exact and Slow-Roll Solutions for Exponential Power-Law Inflation Connected with Modified Gravity and Observational Constraints. Universe 2020, 6, 199.

## The energy scale of inflation

$E_f = |V|^{\frac{1}{4}} = \left|V_o e^{2-\beta\phi}\right|^{\frac{1}{4}} \sim 10^{-2} M_p \sim 10^{16}\text{GeV}$, [8]

Condition $f(t \to \infty) = 0$ leads to $\varphi = 0, H = \lambda, G = G_0, \omega = \omega_0, V = V_0$.

Also, at large times

$$f(T,\varphi) \simeq \alpha_1 \tilde{G}(\varphi) T + \alpha_2 \tilde{V}(\varphi) \simeq -\frac{G_0}{2\sqrt{T_0}} T + \alpha_2 \left(V_0 + \frac{1}{2} G_0 \sqrt{T_0}\right) = -\frac{1}{2} T + \alpha_2 \left(V_0 + \frac{1}{2} G_0\right) = -\frac{1}{2} T - \Lambda,$$

$\Lambda -$ the cosmological constant, $\tilde{G}(\varphi) = \frac{G(\varphi)}{2\sqrt{T_0}}, \tilde{V}(\varphi) = V(\varphi) + T_0 \tilde{G}(\varphi)$,

$T \simeq T_0 = 6H^2 = const$, expanding $\sqrt{T}$ into a series around $T_0$ we obtain

$$\sqrt{T} = \frac{\sqrt{T_0}}{2} + \frac{1}{2\sqrt{T_0}} T + \mathcal{O}[(T - T_0)^2] \,.$$

[8] N. Bellomo, N. Bartolo, R. Jimenez, S. Matarrese and L. Verde, Measuring the Energy Scale of Inflation with Large Scale Structures, JCAP 11 (2018), 043.

# CYCLIC UNIVERSE

$H(t) = \lambda - \mu\alpha\sin(\mu t)\ , f(t) = -\frac{\mu\alpha\sin(\mu t)}{1+\alpha\cos(\mu t)}$, [9]

$a(t) \sim e^{\lambda t}(1 + \alpha\cos(\mu t))$,

$\varphi(t) = \frac{1}{\beta k}\ln\left(1 - \frac{\mu\alpha\sin(\mu t)}{\lambda(1+\alpha\cos(\mu t))}\right)$.

$N(t) = \ln\left(\frac{a}{a_0}\right) = \lambda t + s\ln t$- e-fold number, $a_0$ – the scale factor at the beginning of inflation ($N = 0$).

$f(t) = \frac{s}{t} \Rightarrow H(t) = \lambda + \frac{s}{t}$ – the Hubble parameter,

$t(N)$dependence: $t(N) = \exp\left[\frac{N}{s} - W\left(\frac{\lambda}{s}e^{\frac{N}{s}}\right)\right]$, now we obtain $\varphi(N)$ and $V(N)$:

$\varphi(N) = \frac{1}{\beta k}\ln\left\{1 + \frac{s}{\lambda}\exp\left[W\left(\frac{\lambda}{s}e^{\frac{N}{s}}\right) - \frac{N}{s}\right]\right\}, V(N) = V_0\left\{1 + \frac{s}{\lambda}\exp\left[W\left(\frac{\lambda}{s}e^{\frac{N}{s}}\right) - \frac{N}{s}\right]\right\}^{-\frac{2}{k}}$.

$W$ – the Lambert function.

[9] S. K. Banerjee and J. V. Narlikar, The Quasi-steady state cosmology: A Problem of stability, Astrophys. J. 487, 69 (1997).

# PROPOSED MODEL PROPERTIES

- Different cosmological dynamics does not affect the linear relation $r \sim (1 - n_S)$.
- $f(T, \varphi)_{STG} \sim G(\varphi)\sqrt{T}$ allows one to consider inflationary models with an arbitrary potential of the scalar field and arbitrary cosmological dynamics.
- The main advantage is to obtain the verified cosmological models for an arbitrary model's parameters (including arbitrary dynamics of the expansion of the universe) for the exactly defined type of the scalar-torsion gravity.

# CONCLUSION

- Cosmological models based on scalar-torsion gravity with non-minimal coupling between the scalar field and torsion are considered.
- An inflation models classification according to the expansion order of the tensor-to-scalar ratio dependence on spectral index of the scalar perturbations $r = r(1 - n_S)$ was also proposed. On the basis of this classification, the method for constructing inflationary models based on scalar-torsion gravity verified by observational constraints was considered, implying a linear dependence $r \sim (1 - n_S)$ for arbitrary model's parameters.
- Proposed inflationary models can predict different types of tensor perturbation spectrum (red, blue or flat) depending on the tensor-to-scalar ratio value and how close the early universe cosmological dynamics is to the purely exponential (de Sitter) expansion regime.
- The proposed type of the scalar-torsion gravity implying the wide class of verified cosmological models with arbitrary parameters is of interest for the further deviations research in the relict gravitational waves spectrum and in the compact astrophysical objects evolution from teleparallel equivalent of general relativity or from the other modified gravity theories.

# THANK YOU FOR YOUR ATTENTION!