Should we take into account nontrivial spacetime topology, changes of metric signature and similar hypothetical phenomena when quantizing gravity?

T. P. Shestakova

Department of Theoretical and Computational Physics,

Southern Federal University,

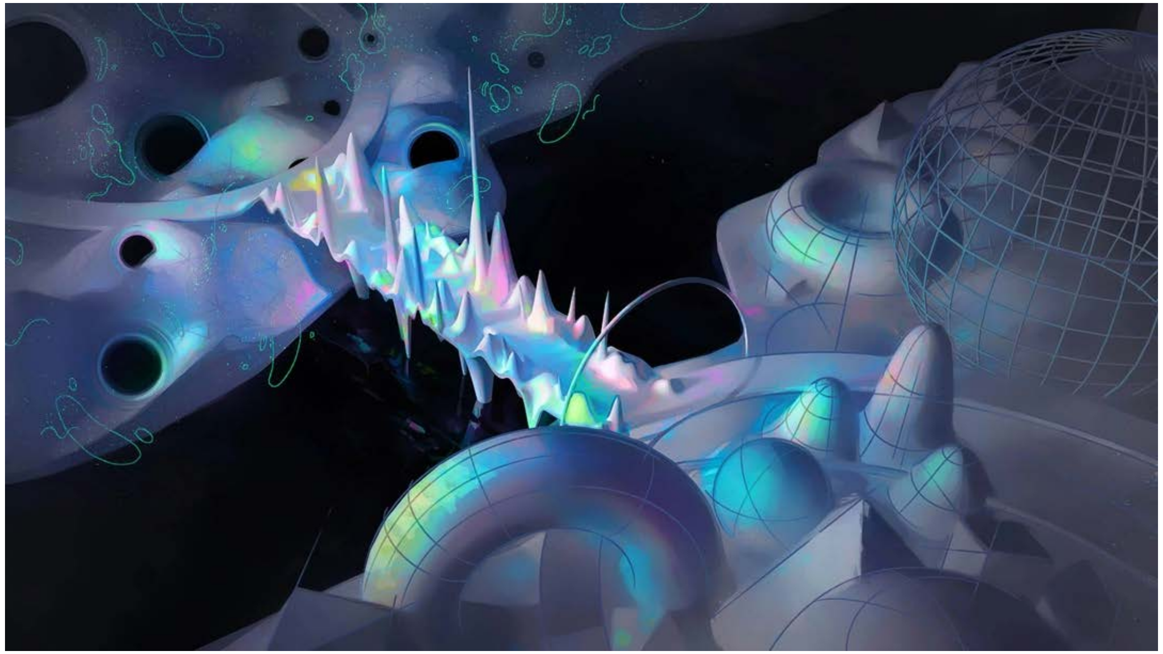
Sorge St. 5, Rostov-on-Don 344090, Russia

E-mail: shestakova@sfedu.ru

Should we take into account nontrivial spacetime topology when quantizing gravity?

- 1955: John Wheeler put forward an idea of fluctuations of spacetime geometry, that later became known as spacetime foam.
- 1979: Stephen Hawking pointed out that one would expect that quantum gravity would allow all possible topologies of spacetime, and it seems that taking into account various topologies may give the most interesting effects.
- 1984: A.D. Sakharov put forward a yet more exotic hypothesis that metric signature may change.

Should we take into account these hypothetical phenomena when quantizing gravity?



The approaches to quantize gravity:

- The canonical approach (based upon Hamiltonian formalism):

$\Longrightarrow$ spacetime topology is restricted by the product of the real line with some three-dimensional manifold, $\mathbb{R} \times \Sigma$.

- The path integral approach:

$\Longrightarrow$ based upon the assumption about asymptotic states, which is true only in asymptotically flat spacetimes.

$\Longrightarrow$ **The both canonical and path integral approaches do not admit an arbitrary spacetime topology.**

What would be if one refuses the assumption about asymptotic states?

- One cannot prove gauge invariance of the path integral and, therefore, gauge invariance of the whole theory.
- The Wheeler – DeWitt equation, which is thought to express this gauge invariance, would lose its sense.

Instead of the Wheeler – DeWitt equation, one can derive a Schrödinger equation

The Schrödinger equation will have a feature of being gauge dependent.

$$i\,\frac{\partial\Psi(N,q,\theta,\bar{\theta};\,t)}{\partial t}=H\Psi(N,q,\theta,\bar{\theta};\,t).$$

$$H=-\frac{1}{2M}\frac{\partial}{\partial Q^{\alpha}}MG^{\alpha\beta}\frac{\partial}{\partial Q^{\beta}}+U(N,q)-V[f]-\frac{1}{N}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}$$

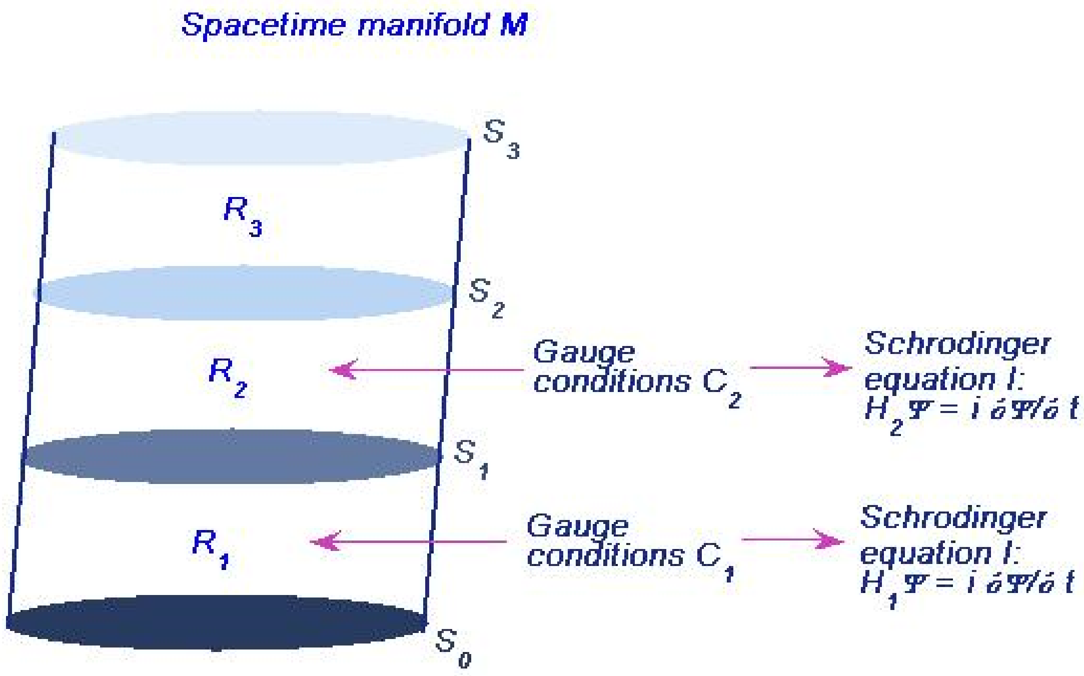
$$G^{\alpha\beta}=\begin{pmatrix} f_{,a}f^{,a} & f^{,a}\\ f^{,a} & g^{ab}\end{pmatrix};\quad Q^{\alpha}=(N,q^{a});$$

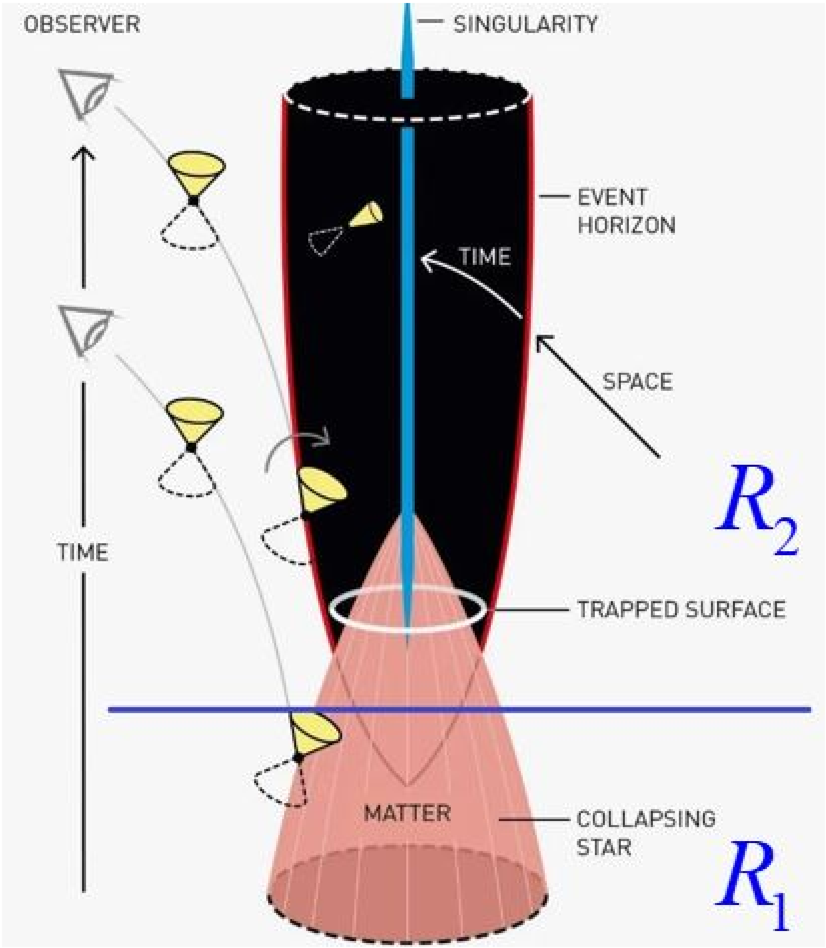
$$\Psi(N,q,\theta,\bar{\theta};\,t)=\int\Psi_{k}(q,t)\,\delta(N-f(q)-k)\,(\bar{\theta}+i\theta)\,dk$$

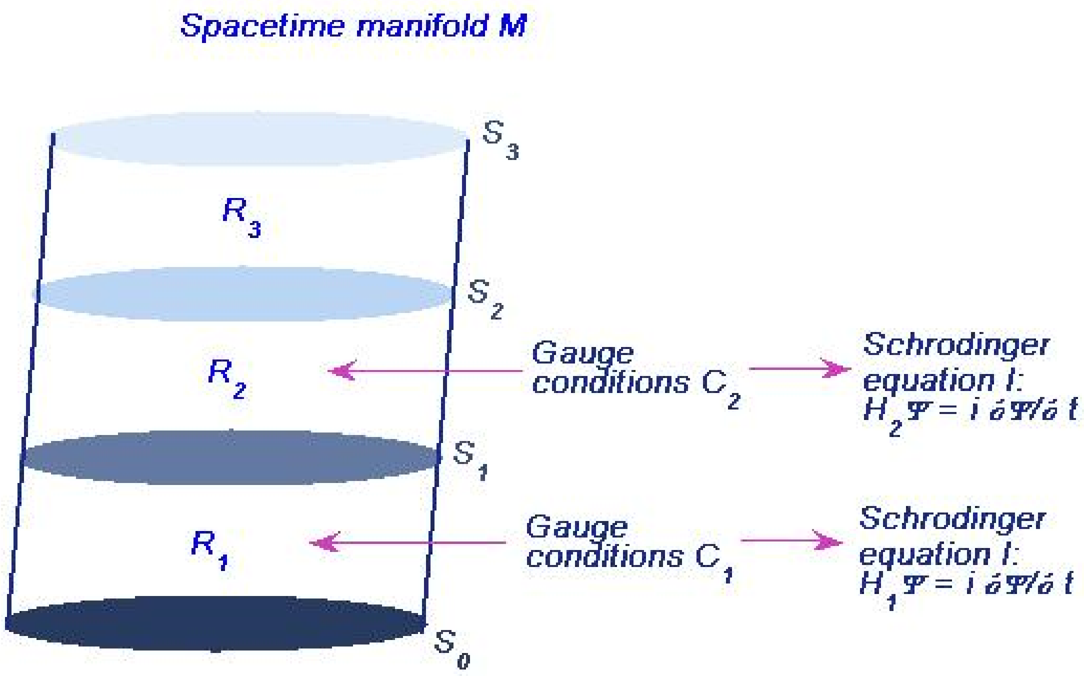
$$i\,\frac{\partial\Psi_{k}(q;\,t)}{\partial t}=H_{(phys)}[f]\Psi_{k}(q;\,t)$$

$$H_{(phys)}[f]=\left.\left(-\frac{1}{2M}\frac{\partial}{\partial q^{a}}Mg^{ab}\frac{\partial}{\partial q^{b}}+U(N,q)-V[f]\right)\right|_{N=f(q)+k}$$

A wave function of the Universe satisfying this equation will describe geometry of the Universe from the point of view of an observer in some fixed reference frame.


Spacetime manifold M
S_3
R_3
S_2
R_2
Gauge conditions C_2
Schrodinger equation I:
$H_2 \Psi = i\, \partial\Psi/\partial t$
S_1
R_1
Gauge conditions C_1
Schrodinger equation I:
$H_1 \Psi = i\, \partial\Psi/\partial t$
S_0


OBSERVER
SINGULARITY
EVENT HORIZON
TIME
SPACE
TIME
R_2
TRAPPED SURFACE
MATTER
COLLAPSING STAR
R_1



$$\left|g_{\mu\nu}^{(1)},\mathcal{S}_1\right\rangle = \exp\left[-iH_{1(phys)}\left(t_1-t_0\right)\right]\left|g_{\mu\nu}^{(0)},\mathcal{S}_0\right\rangle$$

$$\mathcal{P}\left(\mathcal{S}_1,t_1\right)\exp\left[-iH_{1(phys)}\left(t_1-t_0\right)\right]\left|g_{\mu\nu}^{(0)},\mathcal{S}_0\right\rangle$$

$$\left|g_{\mu\nu}^{(3)},\mathcal{S}_3\right\rangle = \exp\left[-iH_{3(phys)}\left(t_3-t_2\right)\right]\mathcal{P}\left(\mathcal{S}_2,t_2\right)\exp\left[-iH_{2(phys)}\left(t_2-t_1\right)\right]\mathcal{P}\left(\mathcal{S}_1,t_1\right)\exp\left[-iH_{1(phys)}\left(t_1-t_0\right)\right]\left|g_{\mu\nu}^{(0)},\mathcal{S}_0\right\rangle$$

At any boundary between the regions with different gauge conditions unitary evolution can be broken down. The operators $\mathcal{P}(\mathcal{S}_i,t_i)$ project the states obtained as a result of unitary evolution in the region R_i onto a basis in the Hilbert space in the neighboring region R_{i+1}.

A small variation of the gauge condition:

$$N = f(q) + k \qquad \Rightarrow \qquad N = f(q) + \delta f(q) + k$$

$$H_{(phys)}[f] = \left(-\frac{1}{2M}\frac{\partial}{\partial q^a}Mg^{ab}\frac{\partial}{\partial q^b} + U(N,q) - V[f]\right)\bigg|_{N=f(q)+k}$$

$$H_{(phys)}[f+\delta f] = \left[-\frac{1}{2M}\frac{\partial}{\partial q^a}Mg^{ab}\frac{\partial}{\partial q^b} + U(N,q) - V[f]\right]\bigg|_{\mu=f(Q^a)+\delta f(Q^a)+k}.$$

$$H_{(phys)}[f+\delta f] = H_{(phys)}[f] + W[\delta f] + V_1[\delta f]$$

$$W[\delta f] = \left[\frac{1}{2M^2}\frac{\partial M}{\partial N}\delta f\frac{\partial}{\partial q^a}Mg^{ab}\frac{\partial}{\partial q^b} - \frac{1}{2M}\frac{\partial}{\partial q^a}\left(\left(\frac{\partial M}{\partial N} - \frac{M}{N^2}\right)\delta f g^{ab}\frac{\partial}{\partial q^b}\right)\right]\bigg|_{\mu=f(Q^a)+k}$$

$W[\delta f]$ is not Hermitian operator with respect to the basis in the region with a gauge condition $N = f(q) + k$.

Time dependent gauge conditions:

The path integral approach implies that one should approximate the effective action, including the gauge condition, at each small time interval $[t_i, t_{i+1}]$.

$$\delta f_i(q) = \alpha f_i(q) \qquad \Rightarrow \qquad N(t) = f(q) + \sum_{i=0}^{n} \alpha f_i(q)\,\theta(t - t_i) + k$$

$$[t_n, t_{n+1}]: \qquad N = f(q) + \sum_{i=0}^{n-1} \alpha f_i(q) + \delta f_n(q) + k$$

In the case of time-dependent gauge condition, it means that at every moment of time we have a Hamilton operator acting in its own "instantaneous" Hilbert space. The "instantaneous" Hamilton operator is a Hermitian operator at every moment of time, but it is non-Hermitian with respect to the Hilbert space that we had at the previous moment.

To summarize, if we admit an arbitrary spacetime topology, it leads to qualitatively new results which were outlined briefly in this talk. Thank you for your attention.