Quartet-metric gravity and scalar graviton dark holes

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In the framework of the modified general relativity theory – the quartet-metric gravity / spontaneously broken relativity, the so called "dark holes" merging a central black hole and a peripheral scalar graviton dark halo are considered. It is shown that the dark holes may naturally explain an effect of asymptotically flat rotation curves, attributed conventionally to the existence of some dark matter. Possibilities of further modification of the basic dark holes, to convert them to more realistic ones for application in astrophysics and cosmology, are discussed.

Outline

- Dark matter (DM) and dark energy (DE) problems are probably the most crucial ones in the modern physics.
  - Strong observational arguments in favor of (cold) DM and DE $\Rightarrow$ the ΛCDM standard cosmological model: the Universe is dominated by DE+DM at present epoch.
  - Problems with particle-like DM: no SM candidate, no preferred BSM model, no signal in direct DM searches, cold DM dynamical problems at small scales.
  - Vacuum energy /cosmological constant (CC) theoretical problems: huge CC from EFT vs tiny observational CC.
- Are (extended) SM + GR sufficient to resolve DM and DE problems?
  An alternative: search for dark degrees of freedom in a modified gravity.
  GR as a Diff gauge EFT: massless tensor graviton with only two polarizations. Where can we get extra degrees of freedom?
- A modified gravity: additional degrees of freedom from spontaneous Diff gauge symmetry breaking. Extra polarizations of (massive) tensor graviton + physical scalar gravi-Higgs boson (scalar graviton): DM and DE candidates.
- An implementation: quartet-metric gravity / spontaneously broken relativity (SBR) [Pirogov 2015] ...
  - The concept, minimal Lagrangian and field equations.
  - Scalar graviton as an effective DM.
  - *Dark holes*: vacuum spherically symmetric stationary solutions.
  - The exceptional solution $\Rightarrow$ **asymptotically flat rotation curves**.
  - Prospects to describe realistic objects: dark hole modifications.

Motivation: DM and DE problems

- The standard ΛCDM model – homogenous, isotropic spatially flat Friedmann–Robertson–Walker Universe dominated (at present) by Dark Energy and Dark Matter: 5% SM matter, 25% (cold) DM (0.5-1.6% 3 SM ν), 70% DE.
  - $\Omega_{tot} = \rho/\rho_c \simeq 1$ from CMB data $\Rightarrow$ spatially flat Universe.
  - SN lumi. distance vs z $\Rightarrow$ accelerated expansion of Universe $\Rightarrow$ $\Omega_\Lambda \simeq 0.7$.
  - **Strong motivation for cold DM:** CMB anisotropy vs present day baryonic density fluctuations $\Rightarrow \Omega_m \simeq 0.3$
- DM is observed in gravitationally bound structures from dwarf galaxies to galaxy clusters: v_r dispersion, rotation curves, gravitational lensing vs visible density.
  - From largest structures: $\langle \rho_{DM} \rangle \simeq 0.264 \rho_c$, consistent with ΛCDM.
  - DM in galaxies: *asymptotically flat rotation curves*, the baryonic Tully–Fisher relation, $v_{rot\,\infty}^{3\div4} \propto M_{bar}$, for $M_{bar} = 10^6 \div 10^{12} M_\odot$ *(DM / SM matter coupling?)*.
- No natural DM candidate, no preferred BSM model providing it, no signal in direct DM searches: accelerator production, galactic WIMP detection ...
  Small scale ($\ll 1$ Mpc) cold DM dynamical problems: • *Missing satellites:* few $M > 10^7 M_\odot$ subhaloes in the MW, much less than expected from simulations. • *The cusp-core problem:* no central DM density peak $\rho(r) \propto 1/r^{0.8\div1.4}$ • *Small DM haloes* of the largest MW satellites ...
- **The vacuum energy (CC) issue:** huge CC from EFT vs tiny observational CC

Problems with CC and particle-like DM $\Rightarrow$ Modified gravity: extra gravitational degrees of freedom, (massive) tensor and scalar gravitons as DM/DE?

Solve DM and DE (small CC) problems in parallel?

A candidate: the quartet-metric gravity / spontaneously broken relativity (SBR) [Pirogov 2015] [Pirogov 2017]

The SBR concept: gravity is described by an EFT built in observer's arbitrary kinematic coordinates x^α on a dynamical metric $g_{\mu\nu}(x^\alpha)$ and a set of distinct dynamical coordinates $z^\alpha \equiv \delta^\alpha_a Z^a(x_\mu)$ where scalar fields Z^a, $a = 0, 1, 2, 3$ are transformed (piecewise in x^α) under the constant Poincaré group acting in Z^a space.

More specifically, the gravity theory is considered as a gauge theory corresponding to a spontaneosly broken relativity symmetry, with Z^a being gravi-Higgs fields. (Such an EFT can be naturally generalized to a spacetime with an arbitrary dimension D.) From particle physics viewpoint, 3 combinations of Z^a components ($D - 1$ at arbitrary D) become additional components of a massive tensor graviton.

The remaining 1 combination of Z^a describes a scalar graviton playing the rôle of a physical gravi-Higgs boson. In what follows, we study manifestations of the latter as a DM [Pirogov 2012]. Structures built of the scalar graviton – *dark holes*, are generalization of GR black holes.

Quartet-metric gravity / SBR [Pirogov 2022]

The EFT of SBR is defined via the generally covariant action functional:

$$S[g_{\mu\nu}, Z^a] = \int \mathcal{L}(g_{\mu\nu}, Z^a) d^4x$$

Four scalar fields $Z^a(x)$, $a = 0, 1, 2, 3$:

- Patchwise invertible: $x^\mu = x^\mu(Z) \Rightarrow$ can use Z^a as distinct coordinates $z^\mu \equiv \delta^\mu_a Z^a(x)$
- Internal symmetry: Lorentz transformations, constant shifts in Z space

$\Rightarrow Z^a$ enter $\mathcal{L}$ only through a quasi-metric $\zeta_{\mu\nu} \equiv \partial_\mu Z^a \partial_\nu Z^b \eta_{ab}$.
$\zeta_{\mu\nu}$ has a (patchwise) inverse $\zeta^{-1\mu\nu}$, $\zeta \equiv \det(\zeta_{\mu\nu}) < 0$. We define an effective scalar field (the scalar graviton or *systolon*):

$$\sigma = \log(\sqrt{-g}/\sqrt{-\zeta}) \,,$$

and tensor fields, the effective metric (i.e. the one defining the observables) $\bar{g}_{\mu\nu}$, and the metric/quasi-metric correlator æ^μ_ν (a kind of dynamical DE):

$$\bar{g}_{\mu\nu} \equiv e^{\bar{w}(\sigma)} g_{\mu\nu}, \ \text{æ}^\mu_\nu \equiv \bar{g}^{\mu\lambda}\zeta_{\lambda\nu} \ \ (\bar{g}^{\mu\nu} \equiv \bar{g}^{-1\,\mu\nu}) \,.$$

In terms of effective fields: $S[g_{\mu\nu}, Z^a] = \int \bar{L}(\bar{g}_{\mu\nu}, \text{æ}^\mu_\nu, \sigma)\sqrt{-\bar{g}}\, d^4x$

The minimal Lagrangian

$$
\begin{aligned}
\bar{L} &= \bar{L}_g + \bar{L}_s + \bar{L}_m \\
\bar{L}_g &= -\frac{1}{2} M_{\rm Pl}^2 \bar{R}(\bar{g}_{\mu\nu}) - \bar{V}_{æ}(\bar{æ}^{\mu}_{\nu}) \\
\bar{L}_s &= \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu s \partial_\nu s - V_s(s)
\end{aligned}
$$

Looking for vacuum solutions only $\Rightarrow$ neglect ordinary matter term L_m.

- Planck mass: $M_{\rm Pl} = 1/(8\pi G_N)^{1/2}$.
- $\bar{R}$ is Ricci scalar curvature.
- $s = M_s \sigma$, $M_s < M_{\rm Pl}$.
- $V_{æ}$ depends on traces of $\bar{æ}^{\mu}_{\nu}$ and is responsible (along with $V_s(s)$) for a spontaneous relativity breaking [Pirogov 2022].
- For the moment, approximate $V_{æ}$ by an effective CC: $\bar{V}_{æ} = M_{\rm Pl}^2 \bar{\Lambda}$, i.e. tensor graviton mass is neglected.

Rewrite $\bar{L}_s$ in terms of σ and factor out its mass scale $M_s \equiv M_{\rm Pl} \Upsilon$:

$$
\bar{L}_s = \bar{L}_\sigma = M_{\rm Pl}^2 \Upsilon^2 \left[\frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V_\sigma(\sigma) \right]
$$

$\Upsilon = M_s / M_{\rm Pl}$ characterizes coupling between scalar and tensor gravity.
$V_\sigma(\sigma) = V_s(\mu\sigma)/M_{\rm Pl}^2 \Upsilon^2$.

The field equations

- $\delta S/\delta g^{\mu\nu} = 0$, $\delta S/\delta Z^a = 0$ (see backup). The latter can be integrated right away. $\Rightarrow$ An integration constant Λ_0 appears, which makes the scalar graviton self-interacting even in case of the Lagrangian $V_\sigma(\sigma) \equiv 0$:

$$V^{\rm eff}(\sigma) = V_\sigma(\sigma) + \frac{\boldsymbol{\Lambda_0}}{\Upsilon^2} e^{-2\bar{w}(\sigma)-\sigma} \,,$$

- The effective energy–momentum tensor for the scalar graviton:

$$\begin{aligned} \bar{T}^{\rm eff}_{\mu\nu} &= \bar{T}_{\sigma\,\mu\nu} + \bar{g}_{\mu\nu} M^2_{\rm Pl} \Lambda_0 e^{-2\bar{w}(\sigma)-\sigma} = \\ &= M^2_{\rm Pl}\Upsilon^2 \left[\partial_\mu\sigma\partial_\nu\sigma - \bar{g}_{\mu\nu} \left(\frac{1}{2}(\partial\sigma)^2 - V^{\rm eff}(\sigma) \right) \right] \,, \end{aligned}$$

- The field equations (cf. (6)–(7) in backup) in familiar form:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{1}{M^2_{\rm Pl}} \bar{T}^{\rm eff}_{\mu\nu} + \bar{g}_{\mu\nu}\bar{\Lambda} \quad (1)$$

$$\frac{1}{\sqrt{-\bar{g}}}\,\partial_\mu \left(\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}\partial_\nu\sigma \right) = -\frac{\partial V^{\rm eff}}{\partial\sigma} \quad (2)$$

(See the equation for Z^a in backup, the two eqs. above are enough for our present purposes.)

Vacuum spherically symmetric stationary dark holes

- To start with, we look for static spherically symmetric vacuum solutions merging a GR-like central black hole and a continuous scalar graviton halo.

The line element in polar coordinates r, θ, ϕ in the reciprocal gauge:

$$ds^2 \equiv \bar{g}_{\mu\nu}\, dx^\mu dx^\nu = A(r)dt^2 - C(r)r^2(\sin^2\theta\, d\phi^2 + d\theta^2) - A^{-1}(r)dr^2$$

$$\text{Rewrite (1) as } \bar{R}^\mu_\nu = \frac{1}{M^2_{\rm Pl}}\left(\bar{T}^{\rm eff}{}^\mu_\nu - \frac{1}{2}\delta^\mu_\nu \bar{T}^{\rm eff}\right) - \delta^\mu_\nu\, 2\bar{\Lambda}$$

$$\uparrow \text{the } \{^0_0\} \text{ component:} \quad \frac{1}{2}\frac{1}{r^2 C}\frac{d}{dr}\left(r^2 C \frac{dA}{dr}\right) = -\Upsilon^2 V^{\rm eff}(\sigma) - \bar{\Lambda} \tag{3}$$

$$\uparrow \text{the } \{^0_0\} - \{^r_r\} \text{ combination:} \quad \frac{1}{r^2 C^{1/2}}\frac{d}{dr}\left(r^2 \frac{dC^{1/2}}{dr}\right) = -\frac{1}{2}\Upsilon^2\left(\frac{d\sigma}{dr}\right)^2 \tag{4}$$

$$\text{Eq. 2 reads:} \quad \frac{1}{r^2 C}\frac{d}{dr}\left(r^2 CA\frac{d\sigma}{dr}\right) = \frac{\partial V^{\rm eff}(\sigma)}{\partial\sigma} \tag{5}$$

$$V^{\rm eff}(\sigma) = V_\sigma(\sigma) + \frac{\Lambda_0}{\Upsilon^2} e^{-2\bar{w}(\sigma)-\sigma}$$

- The scalar halo cannot be "stripped away" unless $V^{\rm eff}(\sigma)$ has a minimum.
- Simplifications (sufficient to find generic features): for the moment we set $V_\sigma \equiv 0$, $\bar{\Lambda} = 0$ and take $\bar{w}(\sigma) = \bar{w}'\sigma$ with a constant $\bar{w}' \neq -1/2$.

If $|1 + 2\bar{w}'| \sim \Upsilon$, the latter is not a small parameter anymore $\Rightarrow$ assume $|1 + 2\bar{w}'| \gg \Upsilon$.

Vacuum spherically symmetric stationary dark holes

- Classify solutions of Eqs. (3)–(5) by Λ_0 sign:
  - $\Lambda_0 < 0$: non-degenerate dark holes [Pirogov 2016]. No analytic parameterization for the entire class of solutions (to our knowledge), still all of them converge at $r \to \infty$ to the exact *exceptional* one featuring **asymptotically flat rotation curves** $\Rightarrow$ gravitational confinment, no flat-space asymptotics (next slide).
  - $\Lambda_0 = 0$: effectively GR with a massless scalar $\Rightarrow$ a degenerate dark hole (cf. Buchdahl–Fisher–Janis–Newman–Winicour solution)
    - Naked singularity with massless scalar hair instead of GR-like black hole. In principle, it can be distinguished from the latter, e.g., by smaller shadow radius / visible mass ratio.
    - Asymptotically flat space at $r \to \infty$.
    - σ is singular at $r \to r_g$, can define central Yukawa charge: $Y_\sigma = 4\pi \int (r^2 CA\sigma')' dr = 4\pi r_g \sigma_r / \sqrt{1 + 2\Upsilon^2 \sigma_r^2}$. Deviations from Schwarzschild metric at $r \gg r_g$ can be arbitrary small if $Y_\sigma \to 0$.

    (details in backup).
  - $\Lambda_0 > 0$: the scalar graviton mimics dynamical DE. Partially studied in a cosmological context [Pirogov 2018]. Considering this in a context of the scalar graviton dark haloes represents an additional interest *(not in this talk)*.

An exact *exceptional* solution, $\Lambda_0 < 0$

(found approximately in [Pirogov 2012] the exact one was obtained in Einstein–Maxwell–dilaton model in [Sheykhi 2006], [Tangen 2007])

Eqs. 3–5 with $V_\sigma = 0$, $\bar{\Lambda} = 0$, $\bar{w}(\sigma) = \bar{w}'\sigma$, $\bar{w}' \neq -1/2$ and $\Lambda_0 < 0$ have an exact three-parametric analytic solution:

$$A(r) = \left(1 - \frac{r_g}{r}\right)\left(\frac{r}{r_h}\right)^{\frac{4\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}}$$

$$C(r) = \left(\frac{r}{r_c}\right)^{-\frac{4\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}}$$

$$\sigma(r) = \frac{2(1+2\bar{w}')}{(1+2\bar{w}')^2+2\Upsilon^2}\log\frac{r}{r_h}$$

- r_g is the Schwarzschild radius of the central black hole.
- The scalar profile parameter:
$$r_h = \Upsilon\left[-\frac{2}{(1+2\bar{w}')^2+2\Upsilon^2}\,\frac{1}{\Lambda_0}\right]^{1/2}$$
- r_c is fixed by the gauge-invariant condition:
$$\frac{d(\text{measured circle length})}{d(\text{measured radius})} \to 2\pi\ ,\ r \to \infty$$
$$\Rightarrow d(rC^{1/2})/(A^{-1/2}dr) \to 1,\ r \to \infty$$
$$\Rightarrow r_c = r_h\left[1 + \frac{2\Upsilon^2}{(1+2\bar{w}')^2}\right]^{\frac{(1+2\bar{w}')^2}{2\Upsilon^2}}$$
($r_c \simeq r_h e$ in the $\Upsilon^2 \ll 1$ limit)
- $Z^a(r)$ fields: see backup

- Scalar/tensor gravity decoupling: $\Upsilon \to 0 \Rightarrow$ Schwarzschild metric.
- Why *exceptional*? • Any solution of Eqs. 3-5 with $V_\sigma = 0$, $\bar{\Lambda} = 0$, $\bar{w}(\sigma) = \bar{w}'\sigma$, $\bar{w}' \neq -1/2$ and $\Lambda_0 < 0$ asymptotically converges to it at $r \to \infty$ [Pirogov 2012].
  • In the non-perturbative $1 + 2\bar{w}' \to 0$ limit, it converges to an effectively 1-dimensional solution rather than to (modified) Buchdahl–Fisher–et al.
- No flat space at $r \to \infty$! The anomalous radial acceleration term $\propto -1/r$.

An exact *exceptional* solution, $\Lambda_0 < 0$ [cross-check in the $C = 1$ gauge →]

- The radial acceleration of a particle at rest:

$$\frac{d^2 r}{ds^2} = -\Gamma^r_{00}\left(\frac{dt}{ds}\right)^2 = -\frac{1}{2}A'(r) = \left[-\frac{r_g}{2r^2} - \frac{2\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}\left(1-\frac{r_g}{r}\right)\frac{\mathbf{1}}{\mathbf{r}}\right]\cdot\left(\frac{r}{r_h}\right)^{\frac{4\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}}$$

- "Geometric" energy/mass of the system (*non-gauge invariant*):

$$M_g(r) = 4\pi M_{\rm Pl}^2\left(\frac{r_c}{r_h}\right)^{\frac{4\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}}\left[\frac{(1+2\bar{w}')^2-2\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}\,r_g + \frac{4\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}\,r\right]$$

The constant term $\propto r_g$ corresponds to the singularity at $r = 0$.
The scalar graviton halo contributes to the divergent term $\propto r$.

- Probing the halo by orbital motion of test particles.

The apparent rotation velocity for a circular orbit with the radius r:

$$v_{rot}(r) = \left[\left(1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}\right)\frac{1}{2}\frac{r_g}{r-r_g} + \frac{\mathbf{2\Upsilon^2}}{(\mathbf{1}+\mathbf{2\bar{w}'})^2}\right]^{1/2}$$

At $\mathbf{r} \to \infty$ $\mathbf{v_{rot}} \to \sqrt{2}\boldsymbol{\Upsilon}/|\mathbf{1}+\mathbf{2\bar{w}'}|$ – an asymptotically flat rotation curve.

Can we model DM haloes seen in galaxies?

A typical asymptotic v_{rot} in large spiral galaxies is ~ 300 km/s (~ 30 km/s in DM dominated dwarfs), thus $\Upsilon \sim 10^{-4} \div 10^{-3} \Rightarrow M_s = \Upsilon M_{\rm Pl} \sim 10^{14\div15}$ GeV *(close to GUT scale?)* A caveat: large deviations from GR-like behaviour – not a problem at $r < r_h$, still need cutoff at $r \gg r_h$ →

Possible modifications and extensions

- The minimal Lagrangian:
  - A class of $\Lambda_0 < 0$ solutions with central "Yukawa charge" $Y_\sigma \neq 0$: an improved description at small distances. *A possibility to describe realistic DM haloes?* [see backup].
  - Rotating dark holes? Cylindrical configurations? Asymptotical behaviour?
  - Approximate DE by an effective CC ($V_æ = \bar{\Lambda}$). Asymptotically de Sitter: $A(r) \simeq 1 - \frac{r_g}{r} + \frac{4\Upsilon^2}{(1+2\bar{w}')^2} \log \frac{r}{r_h} - \frac{\bar{\Lambda}}{3} r^2$. An effective cut off at $r_{cut} \sim \Upsilon / \bar{\Lambda}^{1/2}$ – – *related to DM in galaxy clusters or structures at cosmological scales?*
- Lagrangian modifications:
  - σ self-interaction: $V_{eff}(\sigma) = \frac{\Lambda_0}{\Upsilon^2} e^{-2\bar{w}(\sigma)-\sigma} + V_\sigma(\sigma) = \frac{\Lambda_0}{\Upsilon^2} e^{-2\bar{w}(\sigma)-\sigma} + \bar{\Lambda} + \frac{m^2}{2}(\sigma - \sigma_0)^2 + \dots$. Flat/de Sitter asymptotics? Some numerical examples in backup.
  - Non-minimal σ kinetic term. (Not excluded, as we have an EFT.)
  - Dynamical DE via vacuum/effective metric correlator $æ^\mu_\nu \equiv \bar{g}^{\mu\lambda}\zeta_{\lambda\nu}$, $V_æ(æ^\mu_\nu) \neq 0$. Tensor graviton becomes massive [Pirogov 2022].
- Quasi-stationary and unstable dark holes? QNMs for stationary solutions?
- Matter extended dark holes. Account for baryonic and other SM matter.

  Multi-flavor DM, e.g., σ / heavy ν_s coupling (recall the "Yukawa charge" Y_σ)?
- The effective metric definition: $\bar{g}_{\mu\nu} = e^{\bar{w}(\sigma)} g_{\mu\nu}$. Non-linear $\bar{w}(\sigma)$? Dark / SM matter dependent $\bar{w}(\sigma, \phi_{SM}, \dots)$?
- $\bar{w}(\sigma) = -\sigma/2$: Weyl-transverse gravity. A possibility to resolve the CC / DE problem [Pirogov 2022]. DM aspects require further study.

Summary

This study is an entrance point to a more general problem of merging a modified gravity, DM and DE through the quartet/multiscalar metric gravity / spontaneously broken relativity (SBR):

- We studied basic stationary spherically symmetric vacuum solutions in the SBR framework.
- We identified a class of solutions – *dark holes*, which merge a GR-like central black hole with a continuous scalar graviton halo and asymptotically converge to the exact "exceptional" solution.
- A remarkable manifestation of the dark scalar halo is the effect of *asymptotically flat rotation curves*, similar to the ones observed in galaxies and conventionally attributed to the presence of some DM.
- We briefly discuss possibilities of further modification of basic dark holes with a prospect to describe realistic astrophysical objects.

Just as a black hole is a signature of GR, the *dark hole* may be considered as a signature of the quartet-metric/multiscalar paradigm.
The problem requires further extended study.

Thank you!

Backup

The field equations

back

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S}{\delta \bar{g}^{\lambda\rho}} \frac{\delta \bar{g}^{\lambda\rho}}{\delta g^{\mu\nu}} + \frac{\delta S}{\delta \sigma} \frac{\delta \sigma}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow$$

Einstein's equations (with traceless terms explicitly separated):

$$M_{\rm Pl}^2 \left[\bar{R}_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{R} \right] = \bar{T}_{\sigma\,\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{T}_{\sigma} + $$

$$+ \frac{1}{4} \bar{g}_{\mu\nu} \left[(1 + 2\bar{w}') \left(M_{\rm Pl}^2 (\bar{R} + 4\bar{\Lambda}) + \bar{T}_{\sigma} \right) - \frac{4}{\sqrt{-\bar{g}}} \frac{\delta \sqrt{-\bar{g}} \bar{L}_{\sigma}}{\delta \sigma} \right] \qquad (6)$$

$$\bar{R}_{\mu\nu} = R_{\mu\nu}(\bar{g}_{\mu\nu})$$

$$\bar{T}_{\sigma\,\mu\nu} = M_{\rm Pl}^2 \Upsilon^2 \left[\partial_\mu \sigma \partial_\nu \sigma - \bar{g}_{\mu\nu} \left(\frac{1}{2} (\partial \sigma)^2 - V_{\sigma} \right) \right], \quad \bar{T}_{\sigma} = \bar{g}^{\mu\nu} \bar{T}_{\sigma\,\mu\nu}$$

$$\frac{\delta \sqrt{-\bar{g}} \bar{L}_{\sigma}}{\delta \sigma} = -M_{\rm Pl}^2 \Upsilon^2 \left[\partial_\mu \left(\sqrt{-\bar{g}} \bar{g}^{\mu\nu} \partial_\nu \sigma \right) + \sqrt{-\bar{g}} \frac{\partial V_{\sigma}}{\partial \sigma} \right]$$

$$\bar{w}' = d\bar{w}(\sigma)/d\sigma$$

$$\mathbf{Tr\,(1)} \Rightarrow \frac{1}{4}(1+2\bar{w}') \left(M_{\rm Pl}^2 (\bar{R} + 4\bar{\Lambda}) + \bar{T}_{\sigma} \right) - - \frac{1}{\sqrt{-\bar{g}}} \frac{\delta \sqrt{-\bar{g}} \bar{L}_{\sigma}}{\delta \sigma} = 0$$

The field equations

back

$$\delta S/\delta Z^a = 0 \Rightarrow \partial_\nu \left[\sqrt{-\bar{g}}\, \zeta^{-1\mu\nu} \partial_\mu Z^a \, \frac{1}{4} \left(M_{\rm Pl}^2 (\bar{R} + 4\bar{\Lambda}) + \bar{T}_\sigma \right) \right] = 0$$

Integrate it: without loss of generality, choose $x^\mu \equiv z^\mu \equiv \delta^\mu_a Z^a$, so that $\zeta_{\mu\nu} = \eta_{\mu\nu}$, $\sqrt{-\bar{g}} = e^{2\bar{w}} \sqrt{-g} = e^{2\bar{w}+\sigma} \Rightarrow$

$$\eta^{-1\mu\nu} \delta^a_\mu \, \partial_\nu \left[e^{2\bar{w}+\sigma} \, \frac{1}{4} \left(M_{\rm Pl}^2 (\bar{R} + 4\bar{\Lambda}) + \bar{T}_\sigma \right) \right] = 0 \, .$$

The expression in [...] is a scalar $\Rightarrow$ the following holds in any frame:

$$e^{2\bar{w}+\sigma} \, \frac{1}{4} \left(M_{\rm Pl}^2 (\bar{R} + 4\bar{\Lambda}) + \bar{T}_\sigma \right) = -M_{\rm Pl}^2 \boldsymbol{\Lambda_0} \qquad (7)$$

$M_{\rm Pl}^2 \Lambda_0$ is an arbitrary integration constant playing a crucial rôle.

Degenerate dark holes: $\Lambda_0 = 0$

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At $\Lambda_0 = 0$ Eqs. 1–2 coincide with Einstein–massless scalar equations with the static spherically symmetric solution [Fisher 1948, Buchdahl 1959, Janis–Newman–Winicour 1968]:

$$A(r) = \left(1 - \frac{r_g}{r}\right)^{1/\sqrt{1+2\Upsilon^2\sigma_r^2}}$$

$$C(r) = \left(1 - \frac{r_g}{r}\right)^{1-1/\sqrt{1+2\Upsilon^2\sigma_r^2}}$$

$$\sigma(r) = \frac{\sigma_r}{\sqrt{1+2\Upsilon^2\sigma_r^2}} \log\left(1 - \frac{r_g}{r}\right) + \sigma_0$$

- If $\sigma_r \neq 0$, the $r = r_g$ surface has zero area $\Rightarrow$ naked singularity instead of a black hole. The Schwarzschild metric is recovered in the $\sigma_r \to 0$ limit.
- Minkowski space asymptotics at $r \to \infty$.
- The radial acceleration of a particle at rest (no anomalous acceleration terms decaying slower than $\propto -1/r^2$):

$$\frac{d^2 r}{ds^2} = -\Gamma^r_{00}\left(\frac{dt}{ds}\right)^2 = -\frac{1}{2}A'(r) = -\frac{1}{\sqrt{1+2\Upsilon^2\sigma_r^2}}\left[1 - \frac{r_g}{r}\right]^{1-1/\sqrt{1+2\Upsilon^2\sigma_r^2}} \cdot \frac{r_g}{2r^2}$$

- Define "geometric" mass of the system as $M_g = 2M_{\rm Pl}^2 \int R^0_0 \sqrt{-\bar g} d^3x$ (coinciding in GR with Tolman's energy/mass of a spatially localized system; *non-gauge invariant*):

$$M_g = \int [T^{\rm eff}{}^0_0 - \sum_{i=1,2,3} T^{\rm eff}{}^i_i]\sqrt{-\bar g}\, d^3x = 2M_{\rm Pl}^2 \int R^0_0 \sqrt{-\bar g}\, d^3x = 4\pi M_{\rm Pl}^2 r_g \Big/ \sqrt{1+2\Upsilon^2\sigma_r^2}:$$

- Probing the metric by orbital motion of test particles. For a circular orbit with the radius r: $d\phi/dt = \left[A(r)'/(C(r)r^2)'\right]^{1/2}$. An observer at rest will derive (e.g., from Doppler shifts) a gauge-invariant apparent rotation velocity with account of the gravitational redshift,

$v_{rot}(r) = [C(r)r^2/g_{00}(r)]^{1/2}\, d\phi/dt = \left[(\log A(r))'/(\log C(r)r^2)'\right]^{1/2}$:

$$v_{rot}(r) = \frac{1}{[1+2\Upsilon^2\sigma_r^2]^{1/4}} \cdot \left(\frac{r_g}{2r}\right)^{1/2} \Bigg/ \left[1 - \frac{r_g}{2r}[1 + 1/\sqrt{1+2\Upsilon^2\sigma_r^2}]\right]^{1/2}$$

[cross-check in the $C = 1$ gauge]

The exceptional solution: $Z^a(r)$

$Z^a(r)$ fields in internal polar coordinates
(up to Lorentz transformations and shifts in the internal $\{Z^a\}$ space):

$$Z^0 = C_0 t\,,\; Z^\theta = \theta\,,\; Z^\phi = \phi\,,$$

$$Z^r = r_h\left[\frac{3}{C_0}\frac{(1+2\bar{w}')^2+2\Upsilon^2}{(1+2\bar{w}')(1+6\bar{w}')+2\Upsilon^2}\left(\frac{r_c}{r_h}\right)^{\frac{4\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}}\left(\frac{r}{r_h}\right)^{\frac{(1+2\bar{w}')(1+6\bar{w}')+2\Upsilon^2}{(1+2\bar{w}')^2+2\Upsilon^2}}+C_1\right]^{1/3}$$

$C_0,\; C_1$ are arbitrary constants.
Note that in this particular solution Z^a do not depend on r_g.

back

The exceptional solution: $1 + 2\bar{w}' \to 0$ limit

back

In the $1 + 2\bar{w}' \to 0$ limit the exceptional solution can be, w/o loss of generality, written as:

$$A(r) = A_0 + 2\frac{r_g}{r_h} \cdot \frac{r}{r_h} + \left(\frac{r}{r_h}\right)^2, \ C(r) = \left(\frac{r}{r_c}\right)^{-2}, \ \sigma(r) = const,$$

where $r_h = (-\Lambda_0)^{-1/2}$ and A_0 and r_g are arbitrary parameters. Making a shift $r \to r - r_g$ and redefining $A_0 - (r_g/r_h)^2 \to A_0$, one gets the line element:

$$ds^2 = \left[A_0 + \left(\frac{r}{r_h}\right)^2\right] dt^2 - r_c^2(\sin^2\theta\, d\phi^2 + d\theta^2) - \left[A_0 + \left(\frac{r}{r_h}\right)^2\right]^{-1} dr^2.$$

The spacetime with such a line element can be viewed as a hypercylinder oriented along r axis with the transverse section being a 2D surface of a sphere with the constant radius r_c. In this sense, the exceptional solution in the $1 + 2\bar{w}' = 0$ limit is effectively 1-dimensional.

Note that with $A_0 > 0$ the r coordinate becomes compact as $\int_0^\infty \left[A_0 + \left(\frac{r}{r_h}\right)^2\right]^{-1} dr$ is finite.

This result holds in case of $D > 4$ dimensions, where the transverse section of the hypercylinder becomes an isotropic homogenous $(D-2)$-dimensional space with a constant curvature $1/r_c$.

Modification of the exceptional solution (no Lagrangian modification) back

- The exceptional solution: weak σ singularity at $r \to 0$. The distributed Yukawa charge due to σ self-interaction via $V_{eff}(\sigma)$:

$$Y(r) = 4\pi r^2 C(r)A(r)\sigma'(r) = 4\pi \frac{2(1+2\bar{w}')}{(1+2\bar{w}')^2 + 2\Upsilon^2}(r - r_g)$$

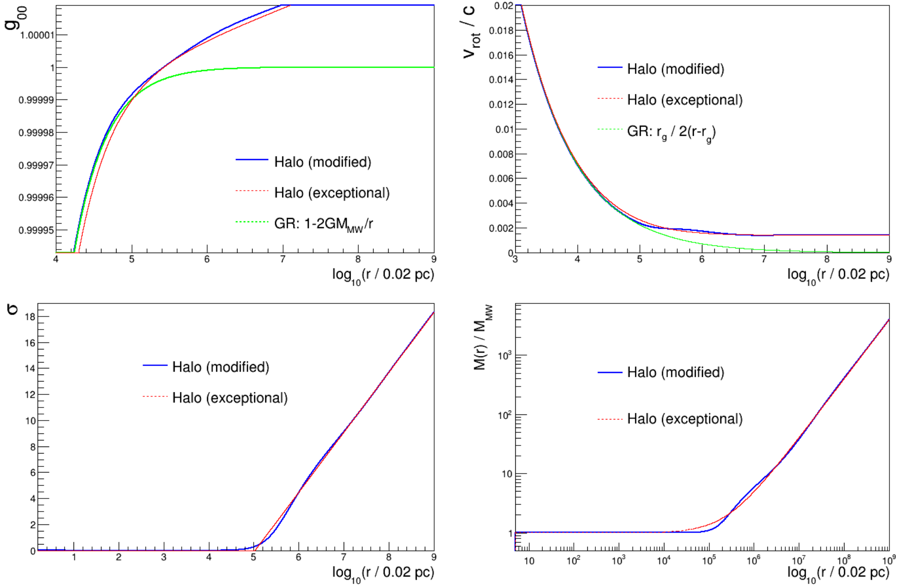
- For *any* solution of Eqs. 3–5 with $\Lambda_0 < 0$, $V_\sigma = 0$, $\bar{\Lambda} = 0$, $\bar{w}(\sigma) = \bar{w}'\sigma$, $\bar{w}' \neq -1/2$:
  - If $\sigma'(r) = 0$ then $\sigma''(r) = -1/A(r) \cdot \Lambda_0 e^{-(1+2\bar{w}')\sigma} > 0 \Rightarrow$ the extremum of $\sigma(r)$ (if exists) is a minimum.
  - **Any solution converges to the exceptional one at $r \to \infty$.** (Consider a finite static deviation from the exceptional solution, show that it tends to 0 at $r \to \infty$ [Pirogov 2012])
  - Any solution is defined by 4 parameters (Start from Λ_0 and A, A', C, C', σ, σ' at some $r = r_0$; $V_{eff}(\sigma) = \Lambda_0 e^{-(1+2\bar{w}')\sigma} \Rightarrow$ can always redefine Λ_0 by $\sigma \to \sigma + \sigma(r_0) \Rightarrow$ -1 parameter; $A(r_0)$ is absorbed into t redefinition, $C(r_0)$ is fixed by the circle length/radius condition $\Rightarrow$ only 4 independent parameters.)

  Note that at $\sigma \to +\infty$ the V_{eff} vanishes: $V_{eff}(\sigma) = \Lambda_0 e^{-(1+2\bar{w}')\sigma} \to 0$ $(1 + 2\bar{w}' > 0)$.

- *Merge degenerate ($Y_\sigma \neq 0$) and exceptional ($Y_\sigma = 0$) solutions:* consider a modified $\Lambda_0 < 0$ solution with a strong $\sigma \to +\infty$ singularity at $r \to r_g$. The $V_{eff}(\sigma) \to 0$ can be neglected $\Rightarrow$ at small r the modified solution behaves like the degenerate one: $\sigma \sim \sigma_r \log(1 - r_g/r) + \sigma_0$, $r \to r_g$, $\sigma_r < 0$, $Y_c \simeq 4\pi r_g \sigma_r$. At $r \gg r_g$ σ reaches its minumum ($\sigma_{min} \simeq \sigma_0$) and converges to the exceptional solution at $r \to \infty$. The intercept point is $r_{int} \simeq r_h e^{\sigma_r/2}$. The modified solution has four parameters: r_g, r_h, Y_c, σ_r (r_c is fixed by the circle length/radius condition at $r \to \infty$). The exceptional solution has $Y_\sigma = 0$ and depends on the two parameters, r_g and r_h.

Modification of the exceptional solution: numerical examples

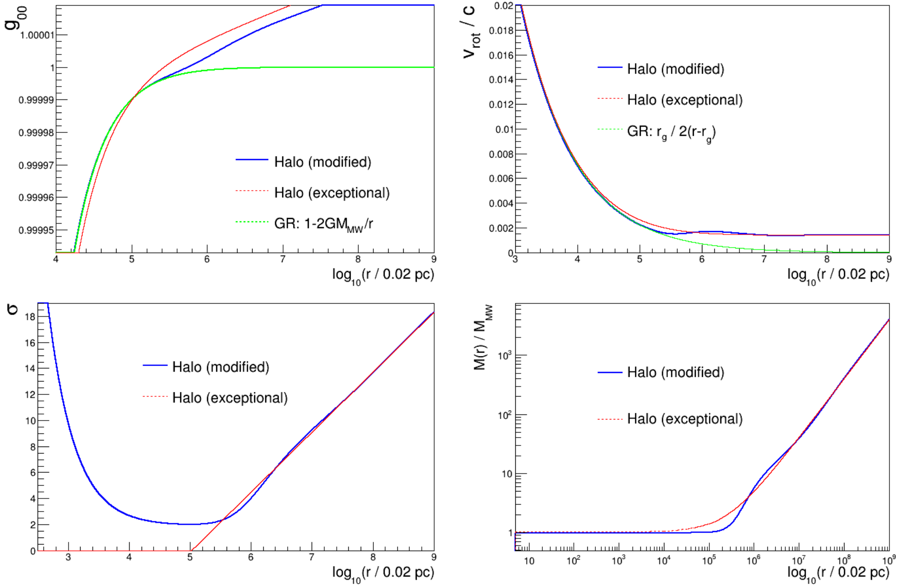
◂ back



$$\Upsilon = 10^{-3},\ r_g = 2GM_{MW}/c^2,\ Y_\sigma/r_g = -0.75,\ \log_{10}\left(r_h e^{\frac{\sigma_r}{2}}/0.02\mathrm{pc}\right) = 5$$

Modification of the exceptional solution: numerical examples

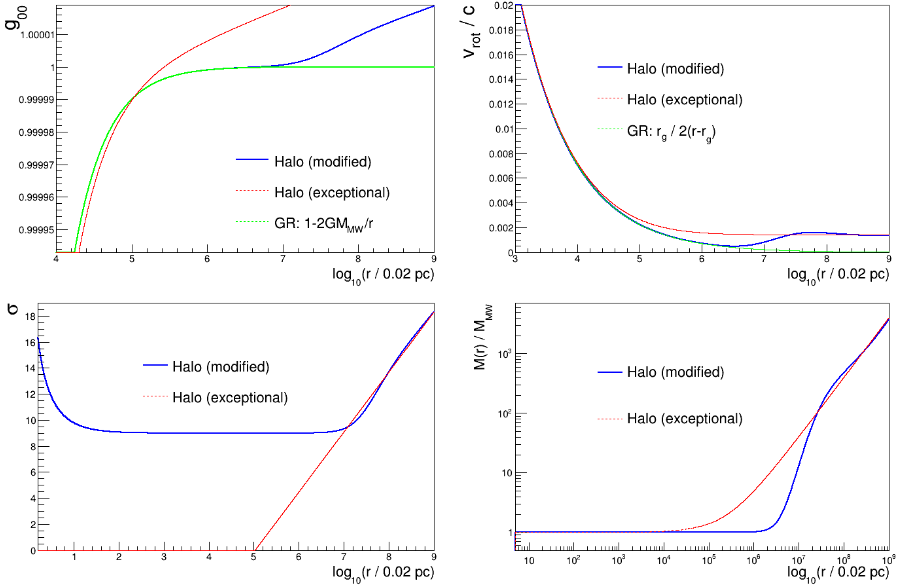
back



$$\Upsilon = 10^{-3},\ r_g = 2GM_{MW}/c^2,\ Y_\sigma/r_g = -0.997\cdot 10^5,\ \log_{10}\left(r_h e^{\frac{\sigma_r}{2}}/0.02\mathrm{pc}\right) = 5.43$$

Modification of the exceptional solution: numerical examples

back



$$\Upsilon = 10^{-3},\ r_g = 2GM_{MW}/c^2,\ Y_\sigma/r_g = -0.93\cdot 10^2,\ \log_{10}\left(r_h e^{\frac{\sigma_r}{2}}/0.02\mathrm{pc}\right) = 6.95$$

Dark hole (Lagrangian) modifications

back

? **How to modify the dark hole to look like a finite DM halo?**

- Need a stationary solution with Minkowski (de Sitter) asymptotics at $r \to \infty$ while keeping the flat rotation curve feature up to cutoff at some r_{cut}.
- All solutions featuring the asymptotically flat rotation curves ($\Lambda_0 < 0$) have non-Minkowski/de Sitter asymptotic metric:

$$ds^2 \sim (r/r_h)^{4\Upsilon^2} dt^2 - r^2 (r/r_h)^{-4\Upsilon^2} d\Omega^2 - (r/r_h)^{-4\Upsilon^2} dr^2.$$

No built-in cutoff mechanism in the minimal Lagrangian.
Some modification possibilities for objects at different scales:

- *Cosmological scale:* recall the CC $\bar{\Lambda} > 0$ (generally, $\bar{V}_{æ} \neq 0$) $\Rightarrow$ an effective halo cutoff at $r_{cut} \sim \Upsilon/\bar{\Lambda}^{1/2} \sim (10^{-4} \div 10^{-2}) \times$ cosmological horizon scale, $r_{cut} \sim 1 \div 100$ Mpc, $M_{halo}(r_{cut}) \sim (10^2 \div 10^4) M_{MW}$ (see $M(r)$ plots)
- *Galactic scale:* $V_\sigma \neq 0$, $V_{eff}(\sigma) = \Lambda_0 e^{-(1+2\bar{w}')\sigma} + V_0 + \frac{m^2}{2}(\sigma - \sigma_0)^2 + \ldots$

$V_{eff}(\sigma)$ has a local minimum if $\frac{-m^2}{(1+2\bar{w}')^2} e^{\sigma_0 - 1/(1+2\bar{w}')} < \Lambda_0 < 0 \Rightarrow \sigma = \sigma_{min} = const$ (quasi-stable) solutions with asymptotically Minkowski (de Sitter) space exist. The $V_{eff}(\sigma)$ minimum is not global $\Rightarrow$ try to interpolate between the (modified) exceptional solution and $\sigma \to \sigma_{min}$?

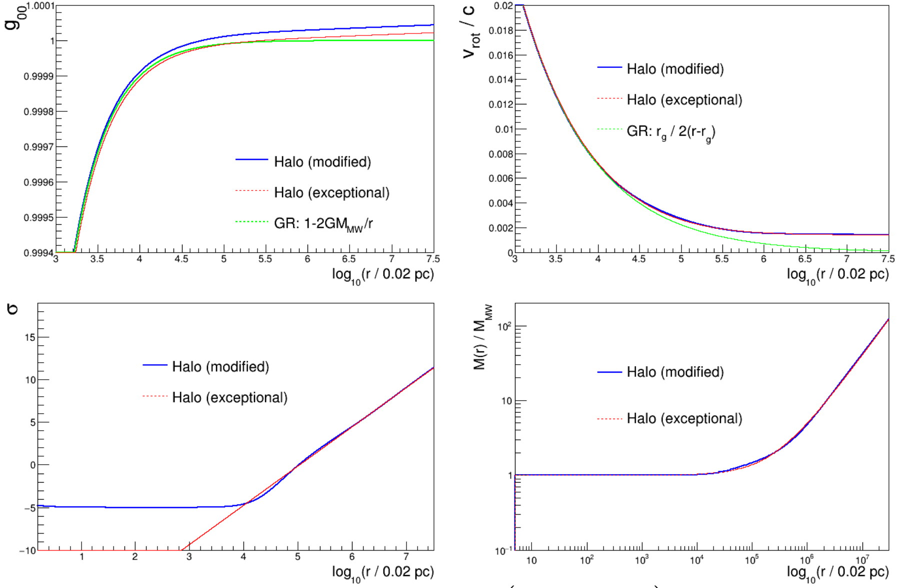
The interpolating solution is 5-parametric (can't shift σ).

The scalar graviton $\delta\sigma = \sigma - \sigma_{min}$ becomes massive in the vicinity of σ_{min}.

Some numerical examples with $V_\sigma \neq 0$ on the next slides →

Modification of the exceptional solution: $V_\sigma = 0$

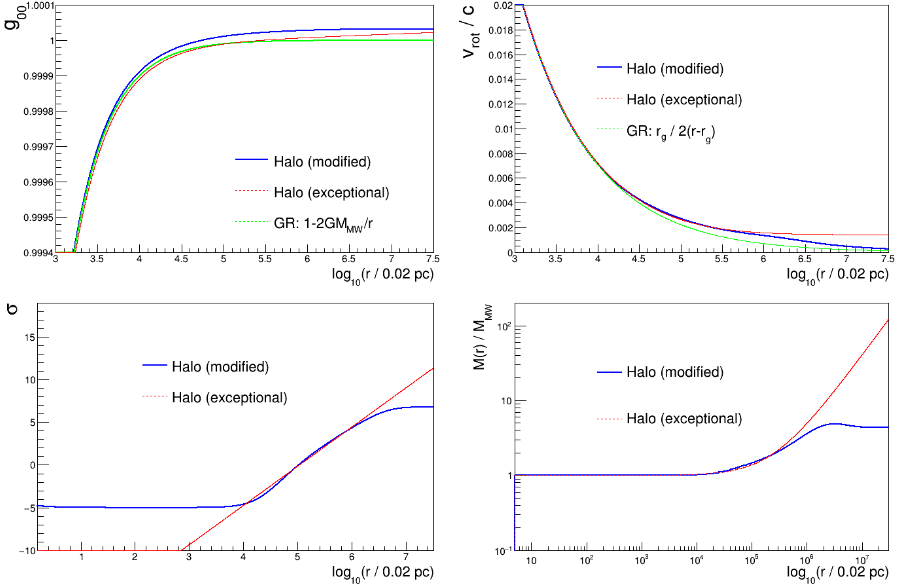
back



$$\Upsilon = 10^{-3},\ r_g = 2GM_{MW}/c^2,\ Y_\sigma/r_g = -3,\ \log_{10}\left(r_h e^{\frac{\sigma_r}{2}}/0.02\mathrm{pc}\right) = 3.91$$

Modification of the exceptional solution: $V_\sigma = m^2/2 \cdot (\sigma - \sigma_0)^2$

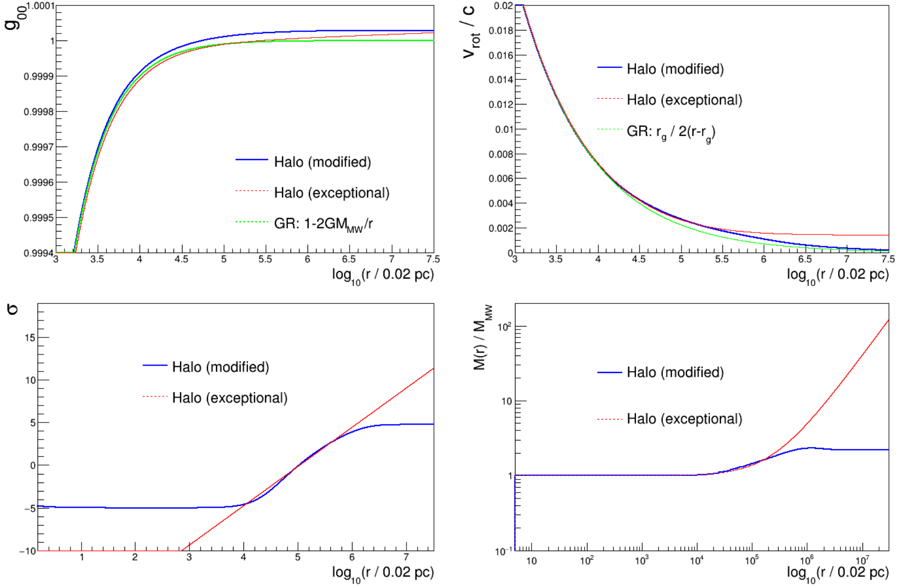
back



The same with $V_\sigma \neq 0$: $\sigma_0 = 7.55$, $1/m = 38.8$ kpc $\Rightarrow$ cutoff at $r_{cut} \simeq 20$ kpc.

Modification of the exceptional solution: $V_\sigma = m^2/2 \cdot (\sigma - \sigma_0)^2$

back



The same with $V_\sigma \neq 0$: $\sigma_0 = 5.55$, $1/m = 14.4$ kpc $\Rightarrow$ cutoff at $r_{cut} \simeq 8$ kpc

Degenerate dark hole: $\Lambda_0 = 0$ ($C = 1$ gauge)

back to the reciprocal gauge

The line element in the $C = 1$ gauge takes the form:

$$ds^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = A(r)dt^2 - r^2(\sin^2\theta \, d\phi^2 + d\theta^2) - B(r)dr^2$$

From Einstein–massless scalar equations 1, 2 one gets the static spherically symmetric solution [Fisher 1948]:

$$A(r) = y^{1-b}, \;\; B(r) = \frac{1}{y}\left(1 - \frac{b}{2} + \frac{b}{2y}\right)^{-2}, \; \sigma(r) = (1-b)\,\sigma_r \log y + \sigma_0 \, ,$$

where $b = 1 - 1/\sqrt{1 + 2\Upsilon^2\sigma_r^2}$ and y satisfies the equation $y = 1 - \frac{r_g}{r} y^{b/2}$.

- "Geometric" gravitating energy/mass of the system reads:

$$M_g = 4\pi M_{\rm Pl}^2 r_g (1-b)$$

- The apparent rotation velocity for a circular orbit with the radius r:

$$v_{rot}(r) = \left[(1-b)\,\frac{r_g}{2r}\,\frac{1}{y^{1-b/2} + b\, r_g/2r}\right]^{1/2}$$

(Coordinate transformation from $C = 1$ to the reciprocal gauge recovers v_{rot} in the reciprocal gauge.)

- The radial coordinate r is related to the one in the reciprocal gauge r_{rec} as $r = r_{rec}(1 - r_g/r_{rec})^{b/2}$. The parameters r_g, σ_r and σ_0 are the same as in the reciprocal gauge.

An exact *exceptional* solution, $\Lambda_0 < 0$ ($C = 1$ gauge)

The line element in the $C = 1$ gauge takes the form:

$$ds^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = A(r)dt^2 - r^2(\sin^2\theta\, d\phi^2 + d\theta^2) - B(r)dr^2$$

From Eqs. 1, 2 one has:

$$A(r) = \left[1 - \left(\frac{r_g}{r}\right)^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}\right]\left(\frac{r}{r_h}\right)^{\frac{4\Upsilon^2}{(1+2\bar{w}')^2}}$$

$$B(r) = \left[1 - \left(\frac{r_g}{r}\right)^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}\right]^{-1}$$

$$\sigma(r) = \frac{2}{1+2\bar{w}'}\log\frac{r}{r_h}$$

The radial coordinate r is related to the one in the reciprocal gauge (r_{rec}) as

$$r^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}} = r_{rec}\cdot r_{c,rec}^{\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}.$$

The same relations for the r_g and r_h parameters:

$$r_g^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}} = r_{g,rec}\cdot r_{c,rec}^{\frac{2\Upsilon^2}{(1+2\bar{w}')^2}},$$

$$r_h^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}} = r_{h,rec}\cdot r_{c,rec}^{\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}$$

The normalization of $B(r)$ is fixed by the circle length asymptotic condition.

- The radial acceleration of a particle at rest:

$$\frac{d^2r}{ds^2} = -\Gamma^r_{00}\left(\frac{dt}{ds}\right)^2 = -\frac{A'}{2AB} = -\frac{r_g}{2r^2}\left[1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}\right]\left(\frac{r_g}{r}\right)^{\frac{2\Upsilon^2}{(1+2\bar{w}')^2}} - \frac{2\Upsilon^2}{(1+2\bar{w}')^2}\left[1-\left(\frac{r_g}{r}\right)^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}\right]\frac{\mathbf{1}}{\mathbf{r}}$$

- The apparent rotation velocity (coordinate transformation to the reciprocal gauge recovers v_{rot} in the latter):

$$v_{rot}(r) = \left[\frac{1}{2}\frac{\left(1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}\right)r_g^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}}{r^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}} - r_g^{1+\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}} + \frac{\mathbf{2\Upsilon^2}}{\mathbf{(1+2\bar{w}')^2}}\right]^{1/2}$$

- "Geometric" energy/mass (*non-gauge invariant*):

$$M_g(r) = 4\pi M_{\rm Pl}^2\left[\left(\frac{r_g}{r_h}\right)^{\frac{2\Upsilon^2}{(1+2\bar{w}')^2}}\left(1-\frac{2\Upsilon^2}{(1+2\bar{w}')^2}\right)r_g + \right.$$
$$\left. + 2\frac{2\Upsilon^2}{(1+2\bar{w}')^2}\left(\frac{r}{r_h}\right)^{\frac{2\Upsilon^2}{(1+2\bar{w}')^2}} r\right]$$