Generalization of Heitler model for electromagnetic cascade

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The 7th International Conference on Particle Physics and Astrophysics (ICPPA-2024)

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QED cascades

- QED cascade (also called electromagnetic cascade) is a chain of successive events of hard photon emission and electron-positron pair photoproduction.
- Cascades arise when a high-energy photon or lepton enters media or strong external field producing a bunch of secondary particles.
- Pair production has a certain energy threshold that limits the number of produced leptons.

Extensive air showers

QED cascades are widely studied as a part of extensive air showers and as a strong-field QED phenomenon.

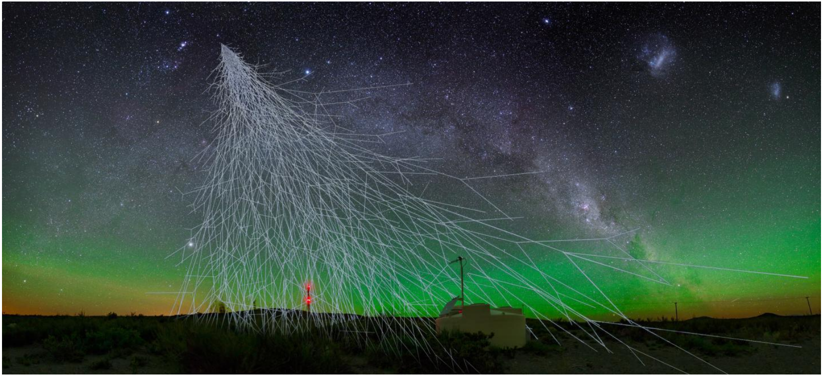

Figure: Visualization of extensive air shower.

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Heitler model for QED cascade

- Energy is split in two in lepton and photon decay processes.
- All processes happen after passing the same free path L.
- Number of leptons N_e in terms of seed electron energy ε_0 and pair producton threshold E_0 at high energy limit $\varepsilon_0 \gg E_0$ can be approximated as

$$N_e \approx \frac{2}{3}\frac{\varepsilon_0}{E_0}. \tag{1}$$

- Depth at which lepton production stops is given by formula

$$t_m \approx L \ln\left(\frac{\varepsilon_0}{E_0}\right). \tag{2}$$

Generalized Heitler model

- Energy is split in two in photon decay processes.
- A fixed fraction k of leptons energy is transfered to an emitted photon, i.e. every lepton with energy ε emits a photon with energy $k\varepsilon$.
- Leptons and photons have different free paths L_e and L_γ, respectively.
- Final number of leptons and cascade depth are to be obtained.

Original and generalized Heitler model comparison

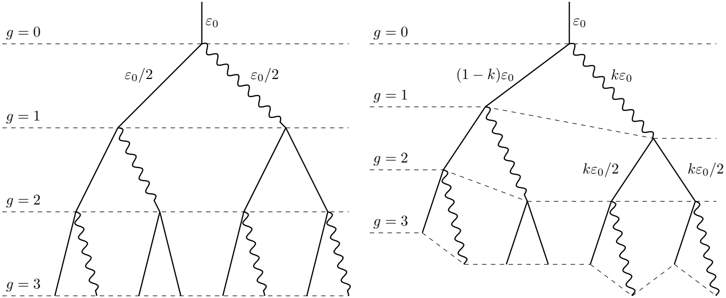

Figure: Schematics of the electron-seeded QED cascade in original (left) and generalized (right) Heitler model. Straight and wavy lines correspond to leptons and photons, respectively. Their different heights represent different free paths, g is the number of generation.

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Basic definitions

- Energy distributions for leptons and photons

$$f_e(\varepsilon, t) = \frac{dN_e(\varepsilon, t)}{d\varepsilon}, \;\; f_\gamma(\varepsilon, t) = \frac{dN_\gamma(\varepsilon, t)}{d\varepsilon}, \tag{3}$$

  where $N_e(\varepsilon, t)$ and $N_\gamma(\varepsilon, t)$ are the numbers of leptons and photons with energies up to ε at depth t, respectively.

- Differential rates $W_{e\to\gamma}(\varepsilon, \varepsilon')$ and $W_{\gamma\to e}(\varepsilon, \varepsilon')$ give the probabilities for photon emission and pair photoproduction per unit depth and unit energy range of the final particle, respectively.
- In case of $W_{e\to\gamma}(\varepsilon, \varepsilon')$, ε and ε' denote the initial lepton energy and the energy of emitted photon, respectively.
- In case of $W_{\gamma\to e}(\varepsilon, \varepsilon')$, ε and ε' are the energies of the initial photon and of the produced electron, respectively.

General form of the equations

- Kinetic equations for QED cascade:

$$\frac{\partial f_e(\varepsilon,t)}{\partial t} = \int_\varepsilon^\infty f_e(\varepsilon',t)\, W_{e\to\gamma}(\varepsilon',\varepsilon'-\varepsilon)d\varepsilon' + \\ + 2\int_\varepsilon^\infty f_\gamma(\varepsilon',t)\, W_{\gamma\to e}(\varepsilon',\varepsilon)d\varepsilon' - \\ - \int_0^\varepsilon f_e(\varepsilon,t)\, W_{e\to\gamma}(\varepsilon,\varepsilon')d\varepsilon',$$

$$\frac{\partial f_\gamma(\varepsilon,t)}{\partial t} = \int_\varepsilon^\infty f_e(\varepsilon',t)\, W_{e\to\gamma}(\varepsilon',\varepsilon)d\varepsilon' - \int_0^\varepsilon f_\gamma(\varepsilon,t)\, W_{\gamma\to e}(\varepsilon,\varepsilon')d\varepsilon'.$$

- Initial conditions for the seeding electron scenario:

$$f_e(\varepsilon,0) = \delta(\varepsilon-\varepsilon_0),\ \ f_\gamma(\varepsilon,0) = 0. \qquad (4)$$

Model equations

- In accordance with our model assumptions we set differential rates

$$W_{e\to\gamma}(\varepsilon,\varepsilon') = \frac{1}{L_e}\delta(\varepsilon' - k\varepsilon),\;\; W_{\gamma\to e}(\varepsilon,\varepsilon') = \frac{1}{L_\gamma}\delta(\varepsilon' - \varepsilon/2). \qquad (5)$$

- Kinetic equations take the form

$$\begin{aligned} \frac{\partial f_e(\varepsilon,t)}{\partial t} =& \frac{1}{L_e\,(1-k)} f_e\left(\frac{\varepsilon}{1-k}, t\right) + \frac{4}{L_\gamma} f_\gamma(2\,\varepsilon, t) - \frac{1}{L_e} f_e(\varepsilon, t),\\ \frac{\partial f_\gamma(\varepsilon,t)}{\partial t} =& \frac{1}{L_e\,k} f_e\left(\frac{\varepsilon}{k}, t\right) - \frac{1}{L_\gamma} f_\gamma(\varepsilon, t). \end{aligned} \qquad (6)$$

- Note that rates (5) and kinetic equations (6) do not include pair production threshold E_0 and are only valid at $\varepsilon \geq E_0$. Accounting for energy threshold would significantly complicate solving the equations and is not necessary for our goals.

Solution for energy distributions

Eqs. (6) can be solved analytically with Mellin transform. Solution for leptons and photons distributions takes form

$$f_e(\varepsilon, t) = \sum_{p,l=0}^{\infty} A_{pl}\,(t, L_e, L_\gamma)\;\delta\big(\varepsilon - \varepsilon_0(1-k)^p(k/2)^l\big), \qquad (7)$$

$$f_\gamma(\varepsilon, t) = \sum_{p,l=0}^{\infty} B_{pl}\,(t, L_e, L_\gamma)\;\delta\big(\varepsilon - \varepsilon_0 k(1-k)^p(k/2)^l\big). \qquad (8)$$

From Eqs. (7) and (8) we can see that the cascade consists of leptons and photons with energies

$$\varepsilon_{pl}^{(e)} = \varepsilon_0(1-k)^p(k/2)^l,\; \varepsilon_{pl}^{(\gamma)} = \varepsilon_0(1-k)^p(k/2)^l k, \qquad (9)$$

The coefficients $A_{pl}(t, L_e, L_\gamma)$ and $B_{pl}(t, L_e, L_\gamma)$ represent the amounts of leptons and photons with the corresponding energies at depth t.

Solution for energy distributions

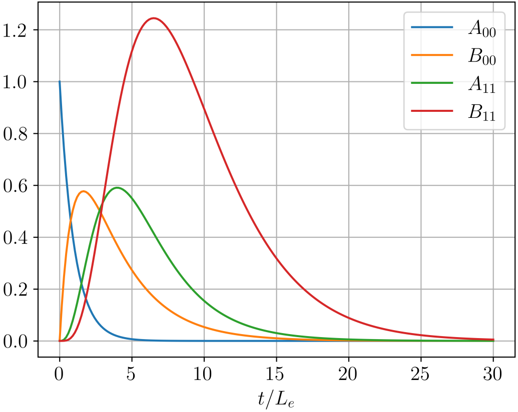

Figure: The dependence of the coefficients A_{00}, B_{00}, A_{11} and B_{11} on depth t for $L_e/L_\gamma = 1/3$.

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General method

- A lepton contributes to the final number of leptons when its energy falls below E_0.
- To track such leptons we count the number of processes with energies of initial particles greater than E_0 and of final leptons lower than E_0.
- Each hard photon emission results in one lepton and each photoproduction results in two leptons.
- General formula reads

$$N_e = \int_0^\infty dt \int_{E_0}^\infty d\varepsilon \int_0^{E_0} [f_e(\varepsilon, t)\, W_{e\to\gamma}(\varepsilon, \varepsilon - \varepsilon') + \\ + \; 2\, f_\gamma(\varepsilon, t)\, W_{\gamma\to e}(\varepsilon, \varepsilon')] d\varepsilon'. \tag{10}$$

Exact formula

By substituting distribution solutions (7) and (8) and rates (5) in general formula (10) we arrive at

$$N_e = \left(\sum_{E_0 \leqslant \varepsilon_0 (1-k)^p (k/2)^l \leqslant E_0/(1-k)} + 2 \sum_{E_0 \leqslant \varepsilon_0 (1-k)^p (k/2)^l k \leqslant 2E_0} \right) 2^l C_{p+l}^l, \quad (11)$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.
Final number of leptons appears to be a function of two dimensionless parameters $\eta = E_0/\varepsilon_0$ and k

$$N_e = N_e(\eta, k). \quad (12)$$

Exact formula

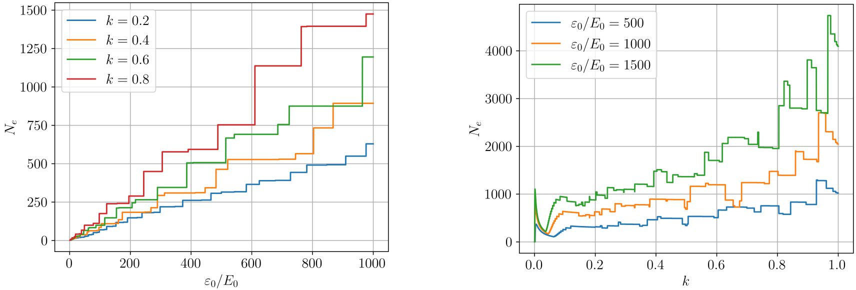

Figure: The final number of leptons as a function of ε_0/E_0 (left) and k (right).

Integral representation

By applying Mellin transform over $\eta = E_0/\varepsilon_0$ to general formula (10) we arrive at

$$\hat{N}_e(s,k) = \int_0^\infty \eta^s N_e(\eta,k) d\eta = \tag{13}$$

$$= \frac{1}{s+1} \frac{1 + 2k^{s+1} - (1-k)^{s+1} - 2(k/2)^{s+1}}{1 - (1-k)^{s+1} - 2(k/2)^{s+1}}. \tag{14}$$

Thus we obtain an integral representation of the final number of leptons as an inverse Mellin transform of (13):

$$N_e(\eta,k) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{ds\ \eta^{-s-1}}{s+1} \frac{1 + 2k^{s+1} - (1-k)^{s+1} - 2(k/2)^{s+1}}{1 - (1-k)^{s+1} - 2(k/2)^{s+1}}, \tag{15}$$

where integral is taken along a vertical line in complex plane and σ is such that $\hat{N}_e(s,k)$ is analytical in the halfplane $\Re(s) > \sigma$.

Integral representation

Function $\hat{N}_e(s,k)$ has singularity points:

- Simple pole at $s = 0$.
- Set of simple poles with $\Re(s) \leq 0$.
- Removable singularity at $s = -1$.

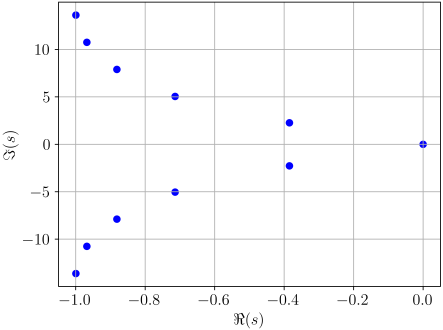

Figure: The location of simple poles of $\hat{N}_e(s,k)$ in the s-plane at $k = 0.2$.

High energy limit

We evaluate the integral in Eq. (15) by choosing any $\sigma > 0$ and using the residue theorem:

$$\sum_{\nu=-\infty}^{+\infty} \frac{1}{s_\nu + 1} \left(\frac{\varepsilon_0}{E_0} \right)^{s_\nu + 1} \frac{1 + 2k^{s_\nu+1} - (1-k)^{s_\nu+1} - 2(k/2)^{s_\nu+1}}{(1-k)^{s_\nu+1} \ln(1/(1-k)) + 2(k/2)^{s_\nu+1} \ln(2/k)}. \tag{16}$$

Since $\Re(s_\nu) \leq 0$, in high energy limit $\varepsilon_0 \gg E_0$ final number of leptons can be written as

$$N_e = N_e^{(1)} + o(\varepsilon_0/E_0), \tag{17}$$

where $N_e^{(1)}$ collects the contribution of all poles with $\Re(s) = 0$ that are linear in ε_0/E_0 and can be used as an approximation.

High energy limit

In the most general case of irrational $\ln(1-k)/\ln(k/2)$, $s=0$ is the only pole with $\Re(s)=0$ and approximate formula for the final number of leptons reads

$$N_e^{(1)} = \frac{\varepsilon_0}{E_0} \frac{2k}{(1-k)\ln(1/(1-k)) + k\ln(2/k)}. \tag{18}$$

In case of rational $\ln(1-k)/\ln(k/2) = m/n$, formula (18) is modified by a factor

$$\frac{\pi e^{\{\pi - \bmod_{2\pi}[y\ln(2\varepsilon_0/E_0)]\}/y}}{y\sinh(\pi/y)}, \tag{19}$$

where $y = -2\pi m/\ln(1-k) = -2\pi n/\ln(k/2)$.

Exact and approximate solutions comparison

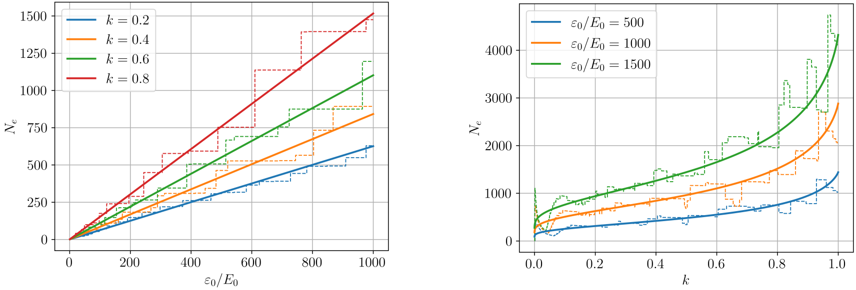

Figure: The final number of leptons (dashed lines) and its linear approximation (solid lines) as functions of ε_0/E_0 (left) and k (right).

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Cascade depth

Depth t_m at which cascade final multiplicity is achieved, i.e. cascade depth, in our model can be roughly estimated

$$t_m \approx \frac{2L_\gamma}{\sqrt{1+8L_\gamma/L_e}-1} \ln\left(\frac{2N_e^{(1)}}{1+1/\sqrt{1+8L_\gamma/L_e}}\right). \tag{20}$$

In case of $k = 1/2$ and $L_e = L_\gamma = L$ the original Heitler model results are reproduced:

$$N_e^{(1)} = \frac{2}{3} 2^{\lceil \log_2(\varepsilon_0/E_0) \rceil}, \tag{21}$$

$$t_m \approx L \ln(2) \left\lceil \frac{\ln(\varepsilon_0/E_0)}{\ln(2)} \right\rceil. \tag{22}$$

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Summary

- A generalized Heitler model for a QED cascade with an arbitrary fixed energy transfer coefficient for photon emission and different free paths of leptons and photons is considered.
- Analytical solutions for distributions above photoproduction threshold are obtained.
- Exact and approximate formulae for the final number of leptons are obtained. At high seeding particle energy the cascade multiplicity is linear in ratio of seeding particle energy to photoproduction threshold.
- A rough estimation for cascade depth in a high energy limit is obtained.
- The original Heitler model results are reproduced as a special case.
- More information can be found on arXiv:2408.06466.

Thank you for your attention!