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Effective potentials in LLA for diverse models in the large N limit

Iakhibbaev R.M., Tolkachev D.M.

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Non-renormalisable models in 4D

The Largangian of general scalar model in diverse dimensions (no higher derivatives)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - g V_0(\phi_i)$$

Examples of potentials:

$$V(\phi) = \lambda \frac{\phi^p}{p!}, g \exp(\phi/m), g \log(\phi/m) \dots$$

Renormalisable potential in **4D**

$$V = \lambda \frac{(\phi \cdot \phi)^2}{4!}$$

Jackiw'74, Itsykson, Martin, Iliopoulos'75

Effective potential

Generating functional

$$Z(J) = \int \mathcal{D}\phi \;\exp\left(i \int d^D x \; \mathcal{L}(\phi, d\phi) + J\phi \right)$$

1PI generating functional and effective action

$$W(J) = -i \log Z(J), \;\; \Gamma(\phi) = W(J) - \int d^D x J(x)\phi(x)$$

Shifted action

$$e^{i\Gamma[\hat{\phi}]} = \int D\phi e^{iS[\phi+\hat{\phi}]-i\hat{\phi}S'[\phi]}$$

Expansion by quantum fields

$$S[\phi + \hat{\phi}] = S[\phi] + \hat{\phi}^2 S''[\phi] + \dots$$

Effective potential

$$V_{eff} = \sum_{k=0}^{\infty} (-g)^k V_k$$

Effective potential in LLA limit

$$V_{eff} = \begin{array}{cl} \begin{pmatrix} V_0 & 0 & \dots \\ a_1\lambda^2 L & \lambda^2 b_1 & \dots \\ a_2\lambda^4 L^2 & \lambda^4 b_2 L & .. \\ \dots & \dots & \dots \\ a_n\lambda^{2n} L^n & b_n\lambda^{2n} L^{n-1} & \dots \end{pmatrix} & \begin{array}{l} \text{Tree-level} \\ \text{One-loop level} \\ \text{Two-loop level} \\ \dots \\ \dots \\ \dots \\ \text{n-loop level} \end{array} \\ \begin{array}{ccc} \textbf{LLA} & \qquad \textbf{NNLA} & \qquad \dots \end{array} & \end{array}$$

Jackiw'74, Itsykson, Martin, Iliopoulos'75

Feynman rules in SO(N)-models

Effective mass

$$m_{ab}^2 = \hat{v}_2 \left(\delta_{ab} - \frac{\phi_a \phi_b}{\phi^2} \right) + v_2 \frac{\phi_a \phi_b}{\phi^2} \qquad m_1^2 = g v_2 = g \frac{\partial^2 V}{\partial \phi^2}, \; m_2^2 = g \hat{v}_2 = 2g \frac{\partial V}{\partial(\phi^2)}$$

Propagators in effective potential method

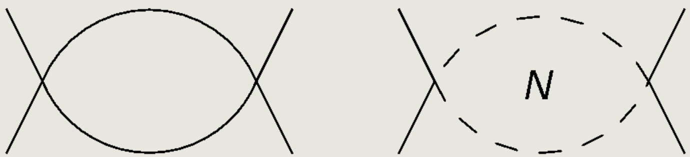
$$G_{1,ab} = \frac{1}{p^2 - m_1^2} \frac{\phi_a \phi_b}{\phi^2} \qquad G_{2,ab} = \frac{1}{p^2 - m_2^2} \left(\delta_{ab} - \frac{\phi_a \phi_b}{\phi^2} \right)$$

Effective vertices

$$v_n = \frac{\partial^n V}{\partial \phi^n}, \; v_n = \frac{\partial^2 V}{\partial \phi^2}, \; \hat{v}_2 = 2 \frac{\partial V}{\partial(\phi^2)} \qquad t_{abcd} = \delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$$

Vacuum diagrams: one-loop

Vacuum one-loop 4D-graphs



$$= \frac{1}{4} v_2^2 \frac{1}{\epsilon} + \frac{1}{4}(N-1)\hat{v}_2^2 \frac{1}{\epsilon}$$

Φ^4-model:

$$\Delta V_1 = g\frac{\phi^4}{4}\frac{1}{4\epsilon} + g(N-1)\frac{\phi^4}{36}\frac{1}{4\epsilon} \quad \rightarrow \frac{g}{64\pi^2}\frac{\phi^4}{4}\log\left(\frac{g\phi^2}{2\mu^2}\right) + (N-1)\frac{g}{64\pi^2}\frac{\phi^4}{36}\log\left(\frac{g\phi^2}{6\mu^2}\right)$$

Φ^6-model:

$$\Delta V_1 = g\left(\frac{\phi^4}{4!}\right)^2\frac{1}{4\epsilon} + g(N-1)\left(\frac{\phi^4}{5!}\right)^2\frac{1}{4\epsilon} \quad \rightarrow \frac{g}{64\pi^2}\left(\frac{\phi^4}{4!}\right)^2\log\left(\frac{g}{\mu^2}\frac{\phi^4}{4!}\right) + \frac{g}{64\pi^2}(N-1)\left(\frac{\phi^4}{5!}\right)^2\log\left(\frac{g}{\mu^2}\frac{\phi^4}{5!}\right)$$

Vacuum diagrams: two-loop

Φ^4 model:

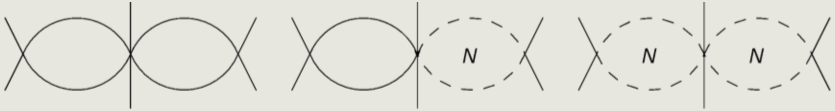
$$\Delta V_2 = \frac{3g^2\phi^4}{32\epsilon^2} + (N-1)\frac{g^2\phi^4}{48\epsilon^2} + (N-1)^2\frac{g^2\phi^4}{864\epsilon^2}$$

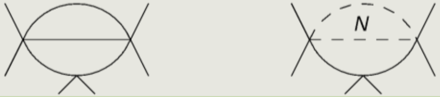
<u>Coincidence</u> with the results of [CC'98, Kastening'96] (even on 3-loop level)

Φ^6 model:

$$\Delta V_2 = (N-1)^2\frac{g^3\phi^{10}}{4^2 5!4!\epsilon^2} + (N-1)\frac{19g^3\phi^{10}}{(5!)^3\epsilon^2} + \frac{7g^3\phi^{10}}{2(4!)^3\epsilon^2}$$

Vacuum two-loop 4D-graphs

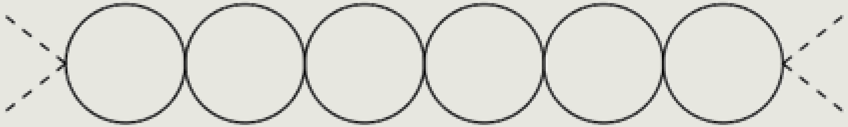




In the large N limit sunset-like diagrams are supressed

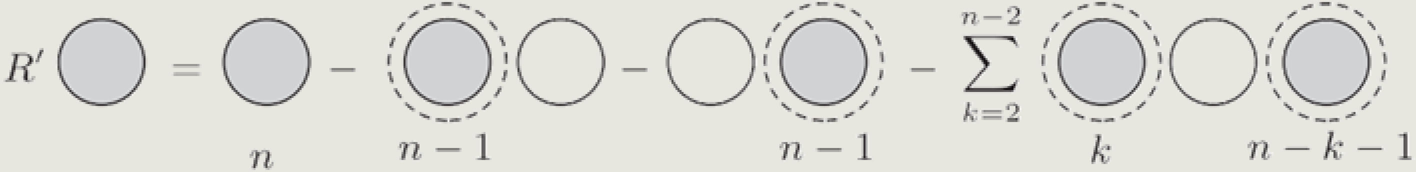
Higher loop diagrams in the large N limit

Φ^4 model:



Other models:

BPHZ and recurrence relation



n-loop divergence **always** is **local** due to Bogoliubov-Parasiuk theorem [BP'57, Hepp'66,Zimmerman'69], result of R'(G) must not contain terms like $\log(\mu^2)/\epsilon$

$$A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n}$$

$$\boxed{\text{Higher order leading divergences are governed by one-loop divergence}}$$

Recurrence relation can be expressed in the next equivalent forms:

$$n\Delta V_n = \frac{N-1}{4}\sum_{k=0}^{n-1}\bar{D}_2\Delta V_k\bar{D}_2\Delta V_{n-k-1}+\frac{1}{4}\sum_{k=0}^{n-1}D_2\Delta V_k D_2\Delta V_{n-k-1} \qquad \bar{D}_2 = 2\frac{\partial}{\partial(\phi^2)}$$

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1}D_{ab}\Delta V_k D_{ba}V_{n-k-1} \qquad D_{ab}\to\frac{\partial^2}{\partial\phi_a\partial\phi_b}$$

RG equation in the large N limit

Recurrence relation in the large N limit

$$n\Delta V_n = \sum_{k=0}^{n-1}\left(\hat{N}\frac{\partial}{\partial(\phi^2)}\Delta V_k \frac{\partial}{\partial(\phi^2)}\Delta V_{n-k-1}\right)$$

RG-PDE

$$\frac{\partial}{\partial z}\Sigma(z,\phi) = -\frac{N}{4}\left(\bar{D}_2\Sigma\right)^2 \qquad \bar{D}_2 = 2\frac{\partial}{\partial(\phi^2)}$$

The equation we study has initial condition

$$\Sigma(0,\phi) = V_0$$

Sum of leading poles

$$\Sigma = \sum_{k=0}(-z)^k \Delta V_k \qquad z = \frac{g}{\epsilon}$$

Effective potential in the large N limit

$$V_{eff} = g\Sigma(z,\phi)\Big|_{z\to\frac{g}{16\pi^2}\log(g\hat{v}_2/\mu^2)}$$

Conditions under which the limit of leading logarithms and the regime of smallness of the coupling constant are satisfied

$$\boxed{\begin{array}{c} g\left(\phi^2\right)^{p/2-2} < 16\pi^2 \\ \log(m(\phi)^2/\mu^2) > 1 \end{array}}$$

Analytic solutions

Introducing ansatz for power-like potential

$$\Sigma(z,\phi) = \frac{(\phi^2)^{p/2}}{p!} f(z(\phi^2)^{p/2-2})$$

PDE turn to ODE

$$-\frac{N}{4p!}\left((p-4)zf'(z) + pf(z)\right)^2 = f'(z) \quad f(0) = 1$$

Introducing ansatz for log-potential

$$\Sigma(z,\phi) = f\left(\frac{g}{(\phi\cdot\phi)^2}\right) - \log(|\phi|/m)$$

PDE turn to ODE

$$\frac{N}{16}\left(1 + 4xf'(x)\right)^2 = -f'(x) \qquad f(0) = 0$$

Introducing ansatz for exp-potential

$$\Sigma(z,\phi_a) = e^{|\phi|/m} f(z/m^4 e^{|\phi|/m})$$

PDE turn to ODE

$$\frac{N}{4}\left(xf'(x) + f(x)\right)^2 = -f'(x) \qquad f(0) = 1$$

Since all these ODE's are the first order differential equations, so they can be solved analytically at least by quadratures

Solutions

Power-like p=4

$$f(z) = \frac{1}{1 + \frac{N}{6} z}$$

Power-like p=6

$$f(z) = \frac{15}{N^3 z^3} \left(90 N z \left(\frac{N z}{30} + 1 \right) - 450 \left(\frac{2 N z}{15} + 1 \right)^{3/2} + 450 \right)$$

Exponential potential

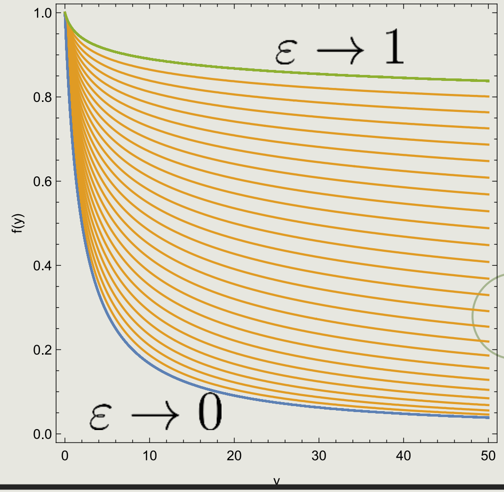
$$f(z) = \frac{W(Nz)(W(Nz) + 2)}{4Nz}$$

Logarithmic potential

$$f(z) = -\frac{\sqrt{Nz+1}}{2Nz} + \frac{1}{2Nz} + \frac{1}{4} \log \left(\frac{4\left(1 - \sqrt{Nz+1}\right)}{Nz\left(\sqrt{Nz+1} + 1\right)} \right) + \frac{1}{4}$$

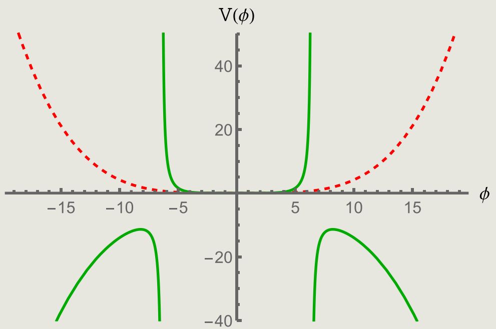
Numerical solutions for

$$p = 4 + \varepsilon$$

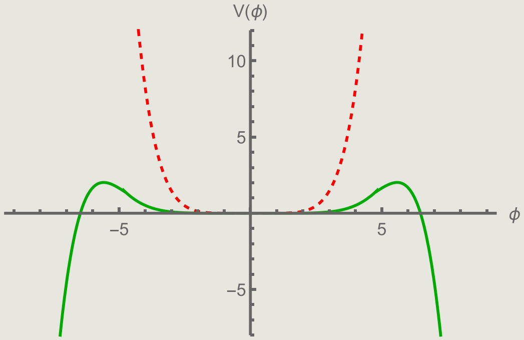


All the above functions have no singularities in the positive region and tend to zero at infinity (zero-charge behaviour), but may have a non-zero imaginary part

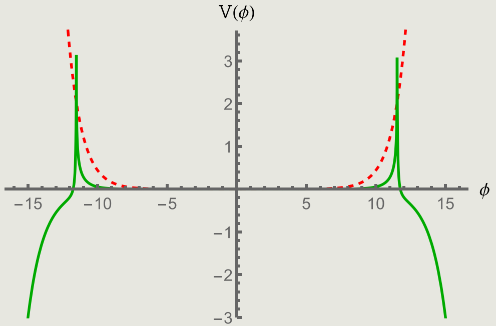
Effective potentials



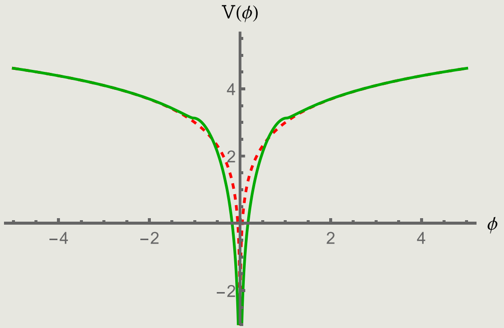
Power-like p=4



Power-like p=6



Exponential potential



Logarithmic potential

All obtained potentials in the leading logarithmic approximation, except for the renormalisable one, are characterised by the metastability of the ground state

Imaginary parts of functions can be identified as decaying rate of metastable state

Conclusions

We calculated

- Feynman rules
- Loop Feynman diagrams up to third order for 4D in arbitrary general scalar models

Possible applications

- Higher dimensions
- Sigma-models
- SUSY-models (Conifold-duals)
- NJL-like models etc

We find

- Iterative relation between leading poles of coefficients
- Generalised RG-equations
- Exact solutions of latters

Theoretical perspectives

- Expansion of analysis on subleading orders and exploring anomalous dimensions
- Studying effective actions
- Scheme-dependence

Thanks

for attention!