Monopole dominance under influence of external magnetic fields

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Introduction

- The monopole is responsible for quark confinement ['tHooft and Mandelstam].
- Instanton is closely related to chiral symmetry breaking [Dyakonov and Shuryak].
- However, the relationship between these phenomena and monopoles and instantons remains unclear.
- Our study, the Pisa group generates gauge field configurations with Nf = 2 + 1 flavor quarks, applying the uniform external magnetic fields.
- To show the effects of the external magnetic fields on the chirl symmetry breaking, we calculate eigenvalues and eigenvectors of overlap fermions and estimate the number of instantons and the hadron masses.

Introduction

- **In previous study,** we found the influence on observables as follows [Theoret. and Math. Phys., 219:3 (2024)]:

  (1) The length of the longest monopole loops increases with increasing the intensity of the external magnetic field.

  (2) The number of instantons and anti-instantons decreases with increasing the intensity of the external magnetic field.

  (3) Chiral symmetry is restored at the external magnetic field around **6 GeV^2**.

- **In this study, we demonstrate Abelian and monopole dominance in the presence of the external magnetic fields.**

- **That is, the numerical results of observables computed from the Abelian and monopole contributions reasonably agree with those computed from the normal configurations without any gauge fixing.**

Gauge fields of Abelian, monopole, and photon contributions

- First, we transform the configurations under the condition of the maximal Abelian gauge [NPB, 190, 3, 455–478 (1981)].
- Then, we define the gauge fields of Abelian, monopole, and photon contributions [NPB 533 576-590 (1998)].
- SU(3) link variable is as follows:

$$\tilde{U}_\mu(s) = C_\mu(s) u_{\mathrm{Ab},\mu}(s)$$

- Definitions of the gauge fields θ_{Ab} and the link variables u_{Ab} of Abelian are as follows:

$$u_{\mathrm{Ab},\mu}(s) = \begin{pmatrix} e^{i\theta^{(1)}_{\mathrm{Ab},\mu}(s)} & 0 & 0 \\ 0 & e^{i\theta^{(2)}_{\mathrm{Ab},\mu}(s)} & 0 \\ 0 & 0 & e^{i\theta^{(3)}_{\mathrm{Ab},\mu}(s)} \end{pmatrix}$$

$$\theta^{(i)}_{\mathrm{Ab},\mu}(s) = arg(\tilde{U}^{(ii)}_\mu(s)) - \frac{1}{3}\phi_\mu(s)$$

$$\phi_\mu(s) = \left[\sum_i arg(\tilde{U}^{(ii)}_\mu(s)) \right]_{\mathrm{mod}2\pi} \in [\pi,\ \pi)$$

Gauge fields of Abelian, monopole, and photon contributions

- Abelian field strength [NPB 533 576-590 (1998)]

$$f_{\mu\nu}^{(i)}(s) = \partial_\mu \theta_{\mathrm{Ab},\nu}^{(i)}(s) - \partial_\nu \theta_{\mathrm{Ab},\mu}^{(i)}(s)$$

- Solving the equation of the motion

$$\theta_{\mathrm{Ab},\mu}^{(i)}(s) = -\sum_{s'} D(s-s')[\partial'_\nu f_{\nu\mu}^{(i)}(s') + \partial_\mu(\partial'_\nu \theta_\nu^{(i)}(s'))]$$

- **D** is the lattice Coulomb propagator.
- We fix the U(1) gauge degree of freedom by imposing the Landau gauge condition.

$$\theta_{\mathrm{Ab},\mu}^{(i)}(s) = -\sum_{s'} D(s-s')\partial'_\nu f_{\nu\mu}^{(i)}(s')$$

- Extracting the monopole (Dirac string) ***n*** from the Abelian field strength to satisfy the following relationship (***k:*** monopole current),

$$\sum_i k_\mu^{(i)} = \sum_i \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial_\nu n_{\rho\sigma}^{(i)} = 0.$$

- Abelian field strength

$$f_{\mu\nu}^{(i)}(s) = \tilde{f}_{\mu\nu}^{(i)}(s) + 2\pi n_{\mu\nu}^{(i)}(s),$$
$$-\pi < \tilde{f}_{\mu\nu}^{(i)}(s) \le \pi,$$
$$n_{\mu\nu}^{(i)}(s) = 0, \pm 1, \pm 2, \pm 3.$$

Gauge fields of Abelian, monopole, and photon contributions

- The gauge fields and the link variables of the monopole and photon contributions are defined as follows [NPB 533 576-590 (1998)]:

$$f_{\mu\nu}^{(i)}(s) = \tilde{f}_{\mu\nu}^{(i)}(s) + 2\pi n_{\mu\nu}^{(i)}(s)$$

$$\theta_{\mathrm{Mo},\mu}^{(i)}(s) = -2\pi \sum_{s'} D(s-s')\partial'_\nu n_{\nu\mu}^{(i)}(s')$$

$$\theta_{\mathrm{Ph},\mu}^{(i)}(s) = -\sum_{s'} D(s-s')\partial'_\nu \tilde{f}_{\nu\mu}^{(i)}(s')$$

$$u_{\mathrm{Ab},\mu}(s) = \begin{pmatrix} e^{i\theta_{\mathrm{Ab},\mu}^{(1)}(s)} & 0 & 0 \\ 0 & e^{i\theta_{\mathrm{Ab},\mu}^{(2)}(s)} & 0 \\ 0 & 0 & e^{i\theta_{\mathrm{Ab},\mu}^{(3)}(s)} \end{pmatrix}$$

$$u_{\mathrm{Mo},\mu}(s) = \begin{pmatrix} e^{i\theta_{\mathrm{Mo},\mu}^{(1)}(s)} & 0 & 0 \\ 0 & e^{i\theta_{\mathrm{Mo},\mu}^{(2)}(s)} & 0 \\ 0 & 0 & e^{i\theta_{\mathrm{Mo},\mu}^{(3)}(s)} \end{pmatrix}$$

$$u_{\mathrm{Ph},\mu}(s) = \begin{pmatrix} e^{i\theta_{\mathrm{Ph},\mu}^{(1)}(s)} & 0 & 0 \\ 0 & e^{i\theta_{\mathrm{Ph},\mu}^{(2)}(s)} & 0 \\ 0 & 0 & e^{i\theta_{\mathrm{Ph},\mu}^{(3)}(s)} \end{pmatrix}$$

Simulation parameters

- The configurations with Nf = 2 + 1 flavor quarks, **physical pion mass**, in SU(3) are generated by the Pisa group [PRD 95, 074515 (2017)].
- The staggered fermion action and the Symanzik tree-level improved gauge action are used.
- The uniform magnetic fields are applied along the **Z** direction (**Bz**).
- The intensity of the uniform external magnetic field varies from **|e|B = 0.57** (Bz = 3) and **1.14** (Bz = 6) [GeV²]: $e|B| = \frac{6\pi Bz}{a^2 L_x L_y}$
- The temperatures T are **50 (low)** and **200** (**high**) [MeV].

V	T [MeV]	a [fm]	Bz	Nconf	Neig
$8^3 \times 16$	50	0.2457	Normal conf, 3, 6	60	200
$8^3 \times 4$	200	0.2457	Normal conf, 3, 6	60	200

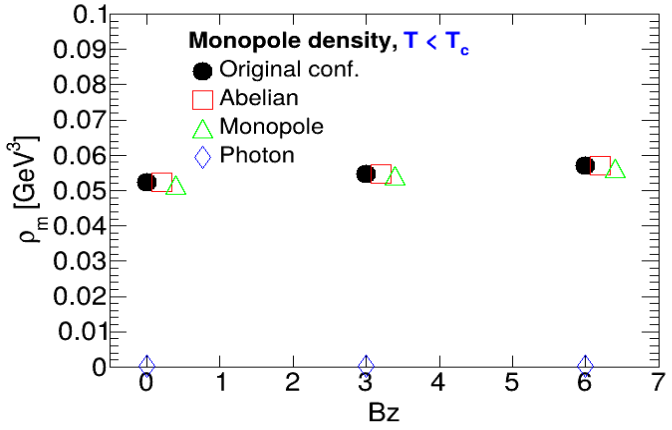
Monopoles in external magnetic fields

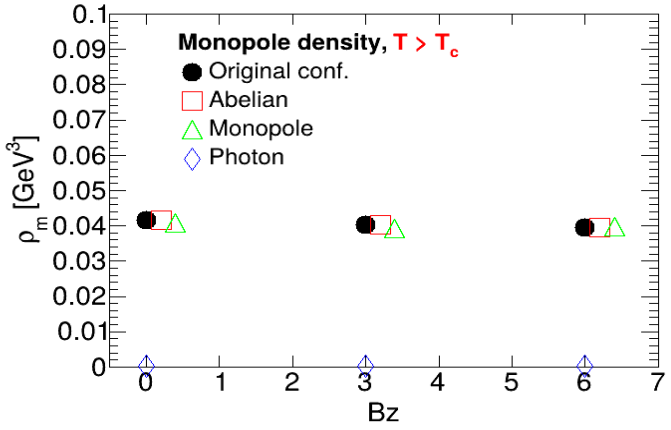
- We perform the gauge transformation of the SU(3) matrices to satisfy the maximal Abelian gauge condition.
- Then, we make the configurations of the Abelian (**Ab**), Monopole (**Mo**), and Photon (**Ph**) gauge fields.
- **First, to confirm the configurations**, we compute the density $\boldsymbol{\rho_m}$ of the monopole current $\boldsymbol{k_\mu}$, which satisfies the current conservation law $\nabla^*_\mu k^i_\mu({}^*n) = 0$. Thus, monopole currents form closed loops [PRD 22 (1980) 2478].
- The definitions of the monopole current, density, and length are as follows:

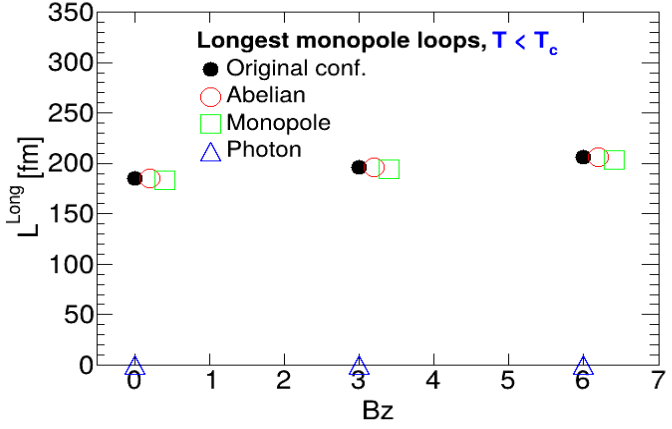
$$k^{(i)}_\mu({}^*n) \equiv -\epsilon_{\mu\nu\rho\sigma}\nabla_\nu n^{(i)}_{\rho\sigma}(n+\hat{\mu}),$$

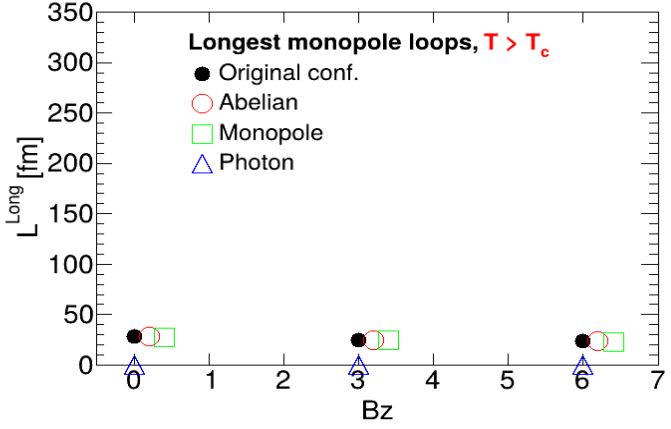
$$\rho_m = \frac{1}{12V}\sum_{i,\mu}\sum_{{}^*n}|k^{(i)}_\mu({}^*n)|/a^3\ [\mathrm{GeV}^3],\ L_m \equiv \frac{a}{12}\sum_{i,\mu}\sum_{{}^*n\in C}|k^{(i)}_\mu({}^*n)|\ [\mathrm{fm}].$$

Monopole density and loops in external magnetic fields









Static potentials of Wilson and Polyakov loops

$T < T_c$ **Wilson loops**

$$V(R) = \log\left[\frac{\langle W(R,T)\rangle}{\langle W(R,T+1)\rangle}\right] = a^2\sigma_{\mathrm{Lat}}R$$

Cornell Potential (fitting function)

$$V(R) = -\frac{\alpha}{R} + \boldsymbol{\sigma} R + V_0$$

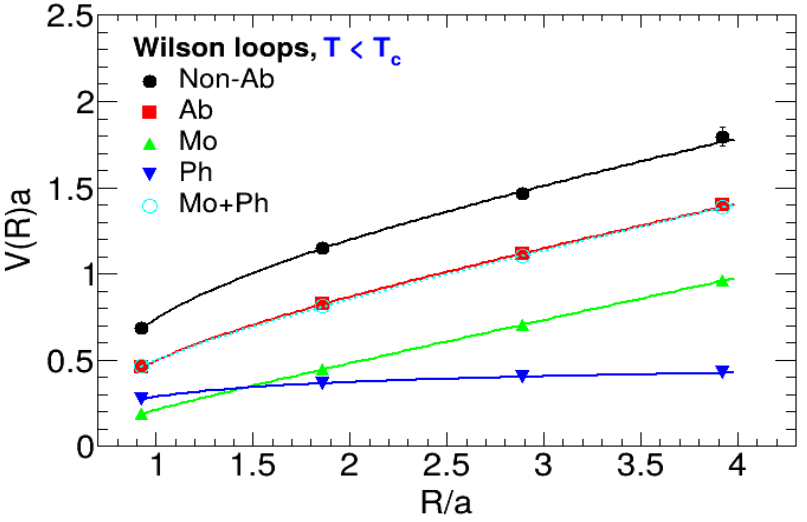
$T > T_c$ **Polykov loops**

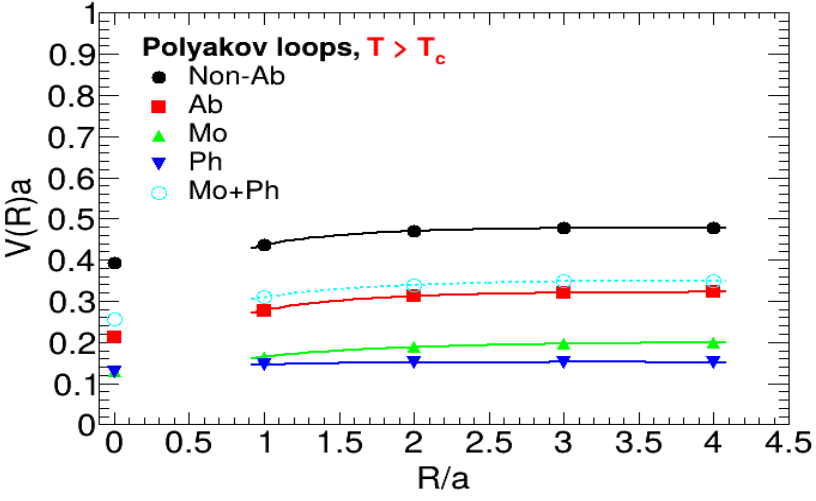
$$V(R) = -T\log\langle P(\vec{s})P^{\dagger}(\vec{s'})\rangle$$

$$R = |\vec{s} - \vec{s'}|$$

Screened Cornell Potential (fitting func.)

$$V(R) = -\frac{\alpha}{R}\exp(-\mu R) + \frac{\boldsymbol{\sigma}}{\mu}(1-\exp(-\mu R)) + V_0$$





Abelian and Monopoles dominance

- **At low temperature, the contributions of the Abelian and monopole dominance to the string tension of Non-Abelian is between 86% and 100%.**
- **At high temperature, the fitting results of the string tension are zero.**

V	Bz	$\frac{\sigma_{\mathrm{Ab}}}{\sigma_{\mathrm{Non\text{-}Ab}}}$	$\frac{\sigma_{\mathrm{Mo}}}{\sigma_{\mathrm{Non\text{-}Ab}}}$
$8^3\times16$	Normal conf	$0.99(13)$	$1.01(12)$
	3	$0.98(13)$	$1.04(12)$
	6	$0.86(11)$	$0.86(10)$

Overlap Dirac operator

- Chiral symmetry: $\mathcal{L} = \bar{\psi} D \psi$

$$\gamma_5 D + D\gamma_5 = 0$$

- The Dirac operator ***D***, which preserves the exact chiral symmetry in the lattice gauge theory, satisfies the following Ginsparg-Wilson relation [P. H. Ginsparg and K. G. Wilson PRD 25 (1982) 2649].

$$\gamma_5 \mathbf{D} + \mathbf{D}\gamma_5 = \mathbf{a}\mathbf{D}\mathbf{R}\gamma_5\mathbf{D}$$

- The following **overlap Dirac operator *D*** satisfies the Ginsparg-Wilson relation [H. Neuberger, PLB 427 (1998) 353]:

$$D(\rho) = \frac{\rho}{a}\left[1 + \frac{D_W(\rho)}{\sqrt{D_W^\dagger(\rho) D_W(\rho)}}\right]$$

$$= \frac{\rho}{a}\{1 + \gamma_5 \epsilon(H_W(\rho))\}$$

- The sign function ***ϵ*** is approximated by the Chebyshev polynomials [Com. Phys. Comm. 153 (2003) 31].
- We solve the eigenvalue problems using **ARPACK**.

$$\mathbf{D}(\rho)|\psi_\mathbf{i}\rangle = \lambda_\mathbf{i}|\psi_\mathbf{i}\rangle$$

- We compute the pairs of eigenvalues $\boldsymbol{\lambda_i}$ and eigenvectors $|\psi_\mathbf{i}\rangle$ from the lowest energy level to approximately 200.

D: Massless overlap Dirac operater.

ϵ: Sign function.

$\mathbf{D_w}$: Massless Wilson Dirac operator.

ρ: Mass parameter **ρ = 1.4.**

$\boldsymbol{H_w}$: Hermitian Wilson Dirac operator.

Zero modes and instantons

- **Fermion zero modes in the eigenvalues.**

  The number of **zero modes** of the **positive chirality is n_+.**

  The number of **zero modes** of the **negative chirality is n_-.**

- The exact zero modes λ_{Zero} are $|\lambda_{\mathbf{Zero}}| \leq \mathbf{O}(10^{-7})$.

- We suppose that the Atiyah-Singer index theorem.

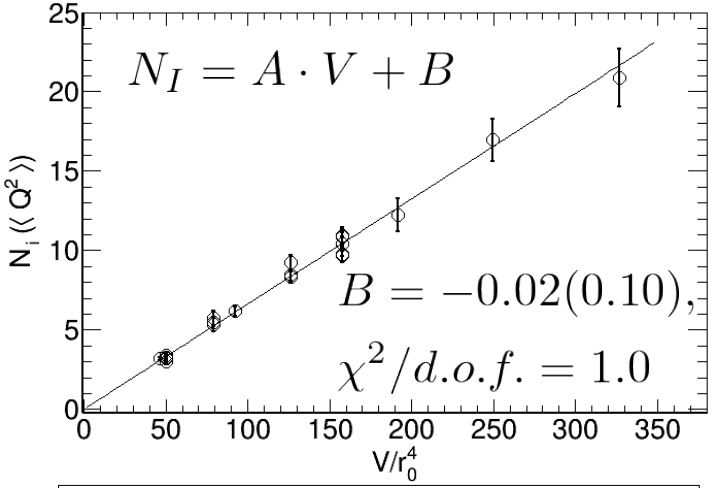
  The number of **instantons** of the **positive charge is n_+.**

  The number of **anti-instantons** of the **negative charge is n_-.**

- We have shown that the number of instantons and anti-instantons N_I can be calculated from the average square of the topological charges Q^2 [PRD 91 (2015) 054512]**:**

$$\mathbf{N_I} = \langle \mathbf{Q^2} \rangle, \ \mathbf{Q} = \mathbf{n_+} - \mathbf{n_-}$$

- **Topological susceptibility is** $\chi = \dfrac{\langle Q^2 \rangle}{V}$.



- **Instanton (anti-instanton) density**

$$\frac{Ar_0^4}{2} = \frac{N_I r_0^4}{2V} = \mathbf{8.09(17)} \times \mathbf{10^{-4}} \ [\mathrm{GeV}^4]$$

- **E. V. Shuryak [NPB 203 (1982) 93]:**

$$\mathrm{n}_c = \mathbf{8} \times \mathbf{10^{-4}} \ [\mathrm{GeV}^4]$$

The number of instantons and anti-instantons

- The number of instantons and anti-instantons N_I is calculated from the average square of the topological charge [PRD 91 (2015) 054512].
  $\mathbf{N_I} = \langle \mathbf{Q^2} \rangle,\ \mathbf{Q} = \mathbf{n_+} - \mathbf{n_-}$
- **The contributions of the Abelian and monopole dominance to the number of instantons and anti-instantons of non-Abelian are between 35% and 68%.**
- **In full QCD with light quarks, the photon contribution affects the property of chiral symmetry.**

V	Bz	$N_I^{\text{Non-Ab}}$	N_I^{Ab}	N_I^{Mo}	$\frac{N_I^{\text{Ab}}}{N_I^{\text{Non-Ab}}}$	$\frac{N_I^{\text{Mo}}}{N_I^{\text{Non-Ab}}}$
$8^3 \times 16$	Normal conf	2.0(3)	0.82(10)	1.30(15)	0.41(8)	0.65(12)
	3	2.0(4)	0.70(16)	1.3(2)	0.36(10)	0.64(16)
	6	1.7 (3)	0.60(15)	1.2(2)	0.35(11)	0.68(16)

Summary and Conclusions

- We investigated the Abelian and monopole dominance over string tension and the number of instantons under the influence of the external magnetic field.
- Even with the effects of the external magnetic field

  (1) At low temperature, the contributions of the Abelian and monopole dominance to the string tension of the standard configuration are between 86% and 100%.

  (2) At high temperature, the fitting results for the string tension are zero.

  (3) At low temperature, the contributions of the Abelian and monopole dominance to the number of instantons and anti-instantons of the standard configurations are between 35% and 68%.

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