New solutions for RG equations for α_s and y_t

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Based on: arXiv:2606.28084 [hep-th]

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Introduction

Callan-Symanzik equation for the Green function Γ:

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} - \gamma(\alpha)\right) \Gamma\left[\frac{Q^2}{\mu^2}, \alpha\right].$$

The solution:

$$\Gamma\left[\frac{Q^2}{\mu^2}, \alpha\right] = \Gamma\left[1, \bar{\alpha}\left(\frac{Q^2}{\mu^2}, \alpha\right)\right] \exp\left[\int_\alpha^{\bar{\alpha}} \frac{\gamma(x)}{\beta(x)} dx\right],$$

where

$$\frac{d\bar{\alpha}}{dL} = \beta(\bar{\alpha}), \quad \bar{\alpha}(0, \alpha) = \alpha, \quad L := \ln(Q^2/\mu^2).$$

Two-coupling theory

One coupling case was examined in `arXiv:2604.04766 [hep-th]`
Standard Model with two couplings: $\alpha := \alpha_s^2/16\pi^2$ and $y := y_t^2/16\pi^2$,
beta-functions (β_{ij} and γ_{ij} are numbers):

$$\frac{d\alpha(L)}{dL} = \beta_0\alpha^2 + \alpha^2\left(\beta_{10}\alpha + \beta_{01}y\right) + \alpha^2\left(\beta_{20}\alpha^2 + \beta_{11}\alpha y + \beta_{02}y^2\right) + \dots$$

$$\frac{dy(L)}{dL} = y\left(\gamma_{10}\alpha + \gamma_{01}y\right) + y\left(\gamma_{20}\alpha^2 + \gamma_{11}\alpha y + \gamma_{02}y^2\right) + y\left(\gamma_{30}\alpha^3 + \gamma_{21}\alpha^2 y + \gamma_{12}\alpha y^2 + \gamma_{03}y^3\right) + \dots$$

$$\alpha(0) = \alpha_0, \quad y(0) = y_0$$

Beta-functions known up to a certain term, we do not solve approximate DE precisely, but rather consider the following series:

$$\alpha = \sum_{\ell=1}^{\infty}\alpha_\ell = \alpha_1 + \alpha_2 + \dots, \quad y = \sum_{\ell=1}^{\infty} y_\ell = y_1 + y_2 + \dots$$

$$\alpha_\ell \sim y_\ell \sim \sum_{n=\ell}^{\infty}(\alpha_0 + y_0)^n L^{n-\ell} \sim (\alpha_0 + y_0)^\ell + (\alpha_0 + y_0)^{\ell+1} L + \dots$$

Linearized equations

$$\alpha_1(0) = \alpha_0,\ y_1(0) = y_0,\ \alpha_\ell(0) = y_\ell(0) = 0\,(\ell > 1)$$

$$\frac{d\alpha_1}{dL} = \beta_0\alpha_1^2,$$

$$\frac{dy_1}{dL} = y_1\left(\gamma_{10}\alpha_1 + \gamma_{01}y_1\right),$$

$$\frac{d\alpha_2}{dL} = 2\beta_0\alpha_1\underline{\alpha_2} + \beta_{10}\alpha_1^3 + \beta_{01}\alpha_1^2 y_1,$$

$$\frac{dy_2}{dL} = \left(\gamma_{10}\alpha_1 + 2\gamma_{01}y_1\right)\underline{y_2} + \gamma_{10}y_1\underline{\alpha_2} + \gamma_{20}\alpha_1^2 y_1 + \gamma_{11}\alpha_1 y_1^2 + \gamma_{02}y_1^3$$

$$\dots$$

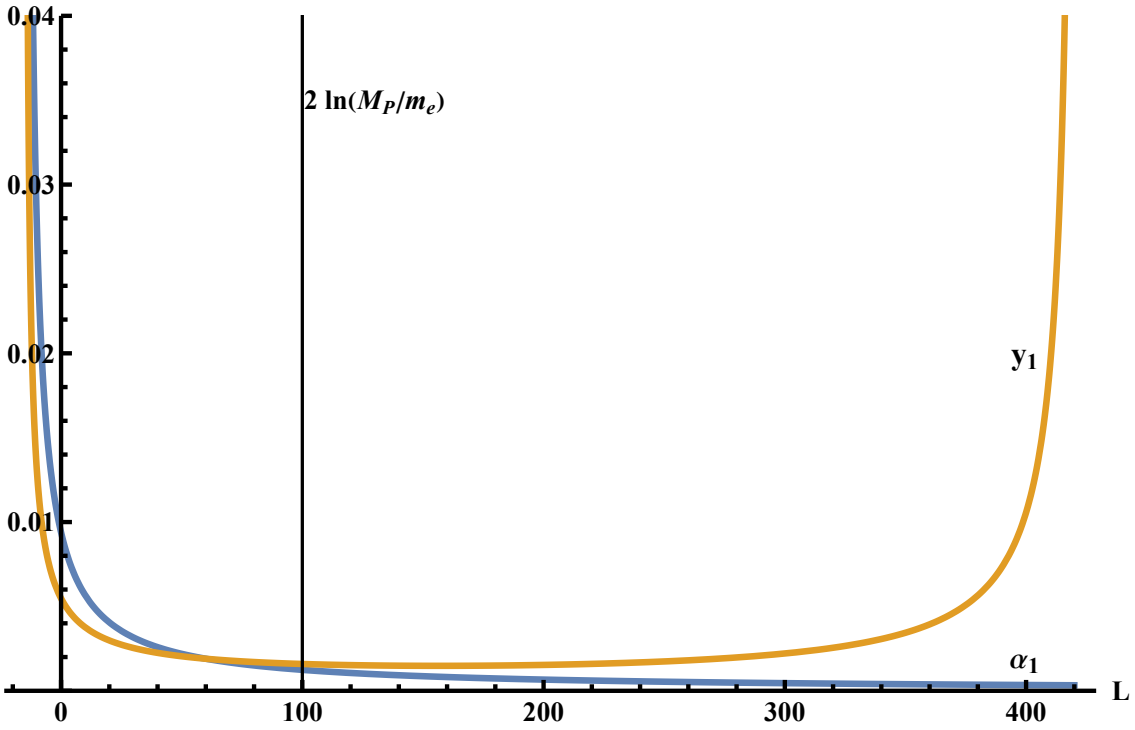
$$\left.\frac{d\alpha_\ell}{dL}\right|_{\ell>1} = 2\beta_0\alpha_1\underline{\alpha_\ell} + \Phi_\ell$$

$$\left.\frac{dy_\ell}{dL}\right|_{\ell>1} = \left(\gamma_{10}\alpha_1 + 2\gamma_{01}y_1\right)\underline{y_\ell} + \gamma_{10}y_1\underline{\alpha_\ell} + \Psi_\ell$$

Φ_ℓ and Ψ_ℓ contain only $\alpha_{\ell'}$ and $y_{\ell'}$ s.t. $\ell' < \ell \Rightarrow$ iterative solving, linear equations $\Rightarrow$ <u>analytical</u> solutions!

$\ell = 1$

$$\begin{cases} \alpha_1 = \dfrac{\alpha_0}{1-\beta_0\alpha_0 L} & \nu = \gamma_{10}/\beta_0 \\ y_1 = \dfrac{(\gamma_{10}-\beta_0)\alpha_0 y_0}{-\gamma_{01} y_0 \left(1-\beta_0\alpha_0 L\right) + \left(\alpha_0\left(\gamma_{10}-\beta_0\right)+\gamma_{01} y_0\right)\left(1-\beta_0\alpha_0 L\right)^{\nu}}. \end{cases}$$



The running of α_1, y_1 in the SM

$\ell = 2$

Equations are linear for $\ell > 1$:

$$\alpha_\ell = \alpha_1^2 \int_0^L \frac{\Phi_\ell}{\alpha_1^2}\, dL',$$
$$y_\ell = \frac{y_1^2}{\alpha_1^\nu} \int_0^L \frac{\alpha_1^\nu}{y_1^2} \left(\gamma_{10} y_1 \alpha_\ell + \Psi_\ell\right)\, dL',$$

$$\alpha_2 = \alpha_2^R = \beta_0 \alpha_1^2 \left(c_1 \ln(\alpha_1/\alpha_0) + c_2 \ln(y_1/y_0)\right),$$

$$y_2 = y_2^R + y_2^+ + y_2^H,$$

$$y_2^R = y_1 \left(\gamma_{10}\alpha_1 + \gamma_{01} y_1\right) \left(c_1 \ln(\alpha_1/\alpha_0) + c_2 \ln(y_1/y_0)\right) + y_1(c_3\alpha_1 + c_4 y_1),$$

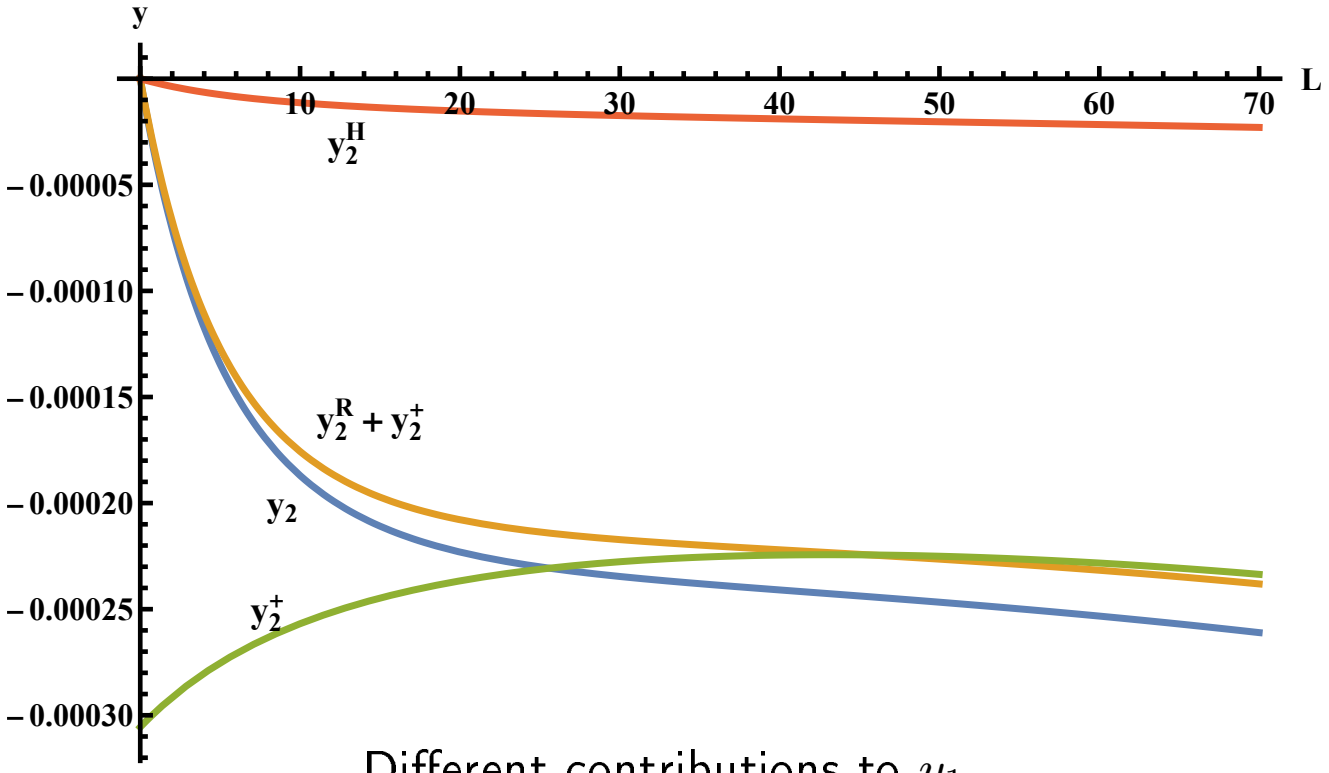
$$y_2^+ = y_1^2 \left(\frac{\alpha_0}{\alpha_1}\right)^{\nu \frac{c_3\alpha_0 + c_4 y_0}{-y_0}},$$

$$y_2^H = c_5 \frac{y_1^2}{\alpha_1^\nu} \int_0^L y_1 \alpha_1^\nu dL'. \qquad \leftarrow \quad \text{Non-elementary function}$$

Analytical solutions!

y_2 structure

$$\frac{d}{dL}y_2^+ = (2\gamma_{01}y_1 + \gamma_{10}\alpha_1)y_2^+, \quad y_2^R(0) + y_2^+(0) = 0$$



Different contributions to y_1

Non-elementary part is insignificant for $y_0 \lesssim \alpha_0$, drop it.

$\ell = 3$

$$
\begin{aligned}
\alpha_3 &= \alpha_3^R + \alpha_3^+ + \alpha_3^H, \\
\alpha_3^R &= \beta_0^2 \alpha_1^3 (c_1 \ln(\frac{\alpha_1}{\alpha_0}) + c_2 \ln(\frac{y_1}{y_0}))^2 \\
&+ \alpha_1^2 \left(\beta_{10}\alpha_1 + \beta_{01} y_1\right) (c_1 \ln(\frac{\alpha_1}{\alpha_0}) + c_2 \ln(\frac{y_1}{y_0})) \\
&+ \beta_0 \alpha_1 (c_6 \alpha_1^2 + c_7 \alpha_1 y_1 + c_8 y_1^2), \\
y_3 &= y_3^R + y_3^+ + y_3^H, \\
y_3^R &= y_1 (\gamma_{01} y_1^2 + \frac{3}{2}\gamma_{10} y_1 \alpha_1 + \frac{\gamma_{10}(\gamma_{10}+\beta_0)}{2\gamma_{01}} \alpha_1^2)(c_1 \ln(\frac{\alpha_1}{\alpha_0}) + c_2 \ln(\frac{y_1}{y_0}))^2 \\
&+ y_1 (c_9 y_1^2 + c_{10} y_1 \alpha_1 + c_{11} \alpha_1^2)(c_1 \ln(\frac{\alpha_1}{\alpha_0}) + c_2 \ln(\frac{y_1}{y_0})) \\
&+ y_1 (c_{12} y_1^2 + c_{13} y_1 \alpha_1 + c_{14} \alpha_1^2).
\end{aligned}
$$

RG-invariant formulation

$\ell = 1$:

$$\hat{\alpha}_1 = \frac{1}{-\beta_0 \hat{L}_1}, \qquad \hat{y}_1 = \frac{\gamma_{10} - \beta_0}{\beta_0 \gamma_{01} \hat{L}_1 + K_1 (\gamma_{10} - \beta_0) \hat{L}^\nu},$$

$$\hat{L}_1 = \ln(Q^2/\Lambda_1^2) = L - \frac{1}{\beta_0 \alpha_0}, \quad K_1 = \frac{\alpha_0(\gamma_{10} - \beta_0) + y_0 \gamma_{01}}{\alpha_0 y_0 (\gamma_{10} - \beta_0)} (-\beta_0 \alpha_0)^\nu.$$

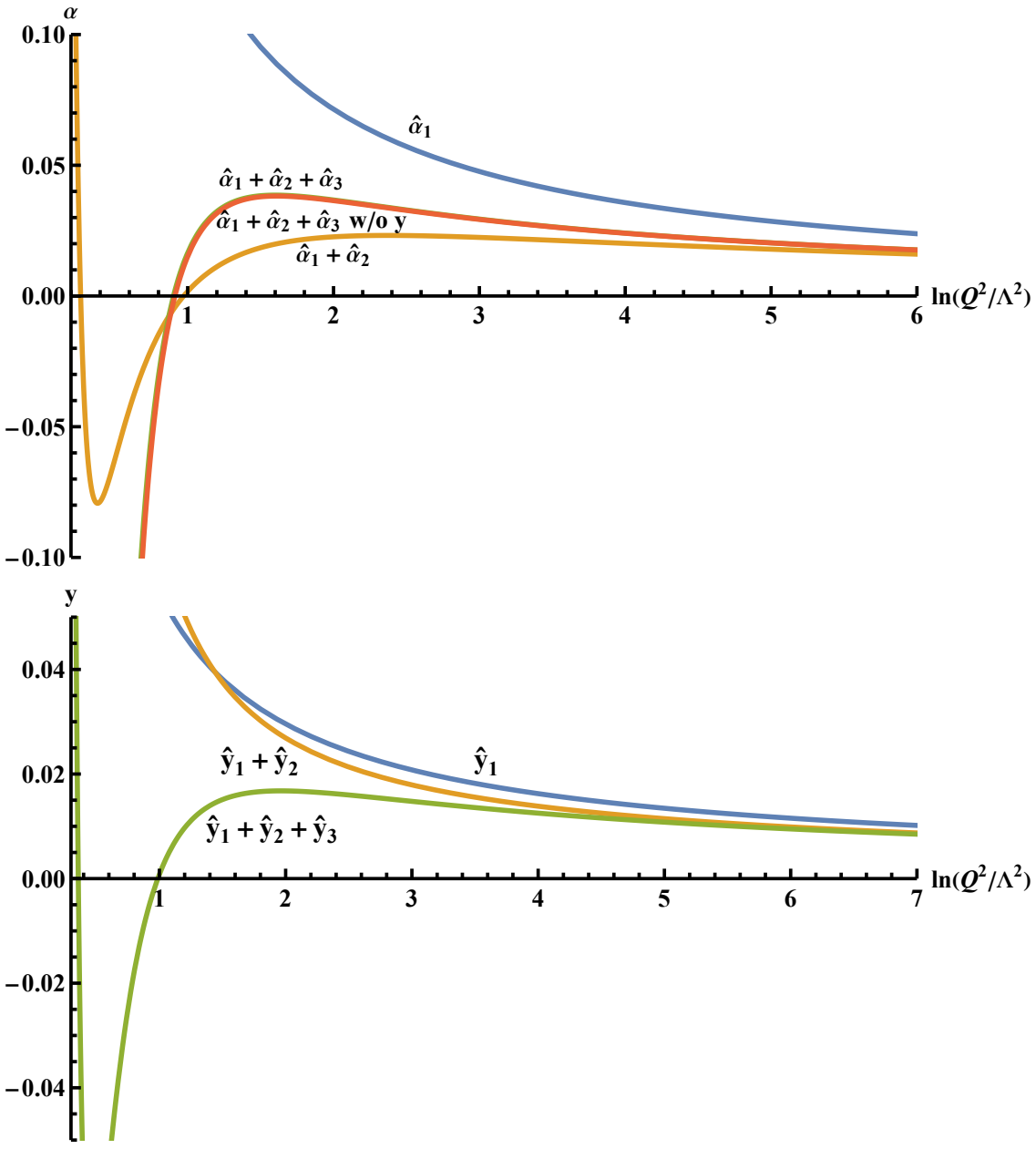
$\uparrow$ Λ_{QCD}

$\uparrow$ new invariant!

Dependence on μ is eliminated, RG-curve is parameterized by two RG-invariants $\hat{L}$ and K:

$$\boxed{\alpha(L, \alpha_0, y_0), y(L, \alpha_0, y_0) \;\to\; \hat{\alpha}(\hat{L}, K), \hat{y}(\hat{L}, K)}$$

RG-invariant solutions



RG-invariant solutions

$$
\begin{aligned}
\hat{\alpha}_1 &= \frac{1}{-\beta_0 \hat{L}}, \\
\hat{\alpha}_2 &= \beta_0 \hat{\alpha}_1^2 \zeta_1, \\
\hat{\alpha}_3 &= \beta_0^2 \hat{\alpha}_1^3 \zeta_1^2 + \hat{\alpha}_1^2 (\beta_{10} \hat{\alpha}_1 + \beta_{01} \hat{y}_1) \zeta_1 + \beta_0 \hat{\alpha}_1 (c_5 \hat{\alpha}_1^2 + c_6 \hat{\alpha}_1 \hat{y}_1 + c_7 \hat{y}_1^2), \\
\hat{y}_1 &= \frac{\gamma_{10} - \beta_0}{\beta_0 \gamma_{01} \hat{L} + K (\gamma_{10} - \beta_0) \hat{L}^\nu}, \\
\hat{y}_2 &= \hat{y}_1 (\gamma_{10} \hat{\alpha}_1 + \gamma_{01} \hat{y}_1) \zeta_1 + \hat{y}_1 (c_3 \hat{\alpha}_1 + c_4 \hat{y}_1), \\
\hat{y}_3 &= \hat{y}_1 (\gamma_{01} \hat{y}_1^2 + \frac{3}{2} \gamma_{10} \hat{y}_1 \hat{\alpha}_1 + \frac{\gamma_{10} (\gamma_{10} + \beta_0)}{2 \gamma_{01}} \hat{\alpha}_1^2) \zeta_1^2 \\
&+ \hat{y}_1 (c_8 \hat{y}_1^2 + c_9 \hat{y}_1 \hat{\alpha}_1 + c_{10} \hat{\alpha}_1^2) \zeta_1 + \hat{y}_1 (c_{11} \hat{y}_1^2 + c_{12} \hat{y}_1 \hat{\alpha}_1 + c_{13} \hat{\alpha}_1^2),
\end{aligned}
$$

$\ldots,$

$$\zeta_1 := c_1 \ln \hat{\alpha}_1 + c_2 \ln \hat{y}_1.$$

Series of leading, subleading, etc series of logarithms of couplings (powers of ζ_1) reproduce solutions α_ℓ and y_ℓ!

RG-invariant solutions

$$
\begin{aligned}
\hat{\alpha}_1 &= \frac{1}{-\beta_0 \hat{L}}, \qquad \text{1st vertical series} \\
\hat{\alpha}_2 &= \beta_0 \hat{\alpha}_1^2 \zeta_1, \\
\hat{\alpha}_3 &= \beta_0^2 \hat{\alpha}_1^3 \zeta_1^2 + \hat{\alpha}_1^2 (\beta_{10} \hat{\alpha}_1 + \beta_{01} \hat{y}_1) \zeta_1 + \beta_0 \hat{\alpha}_1 (c_5 \hat{\alpha}_1^2 + c_6 \hat{\alpha}_1 \hat{y}_1 + c_7 \hat{y}_1^2), \\
\hat{y}_1 &= \frac{\gamma_{10} - \beta_0}{\beta_0 \gamma_{01} \hat{L} + K (\gamma_{10} - \beta_0) \hat{L}^\nu}, \\
\hat{y}_2 &= \hat{y}_1 (\gamma_{10} \hat{\alpha}_1 + \gamma_{01} \hat{y}_1) \zeta_1 + \hat{y}_1 (c_3 \hat{\alpha}_1 + c_4 \hat{y}_1), \\
\hat{y}_3 &= \hat{y}_1 \left(\gamma_{01} \hat{y}_1^2 + \frac{3}{2} \gamma_{10} \hat{y}_1 \hat{\alpha}_1 + \frac{\gamma_{10} (\gamma_{10} + \beta_0)}{2 \gamma_{01}} \hat{\alpha}_1^2\right) \zeta_1^2 \\
&+ \hat{y}_1 (c_8 \hat{y}_1^2 + c_9 \hat{y}_1 \hat{\alpha}_1 + c_{10} \hat{\alpha}_1^2) \zeta_1 + \hat{y}_1 (c_{11} \hat{y}_1^2 + c_{12} \hat{y}_1 \hat{\alpha}_1 + c_{13} \hat{\alpha}_1^2), \\
&\dots,
\end{aligned}
$$

$$
\zeta_1 := c_1 \ln \hat{\alpha}_1 + c_2 \ln \hat{y}_1 .
$$

Series of leading, subleading, etc series of logarithms of couplings (powers of ζ_1) reproduce solutions α_ℓ and y_ℓ!

RG-invariant solutions

$$
\begin{aligned}
\hat{\alpha}_1 &= \frac{1}{-\beta_0 \hat{L}}, \qquad \text{2nd vertical series} \\
\hat{\alpha}_2 &= \beta_0 \hat{\alpha}_1^2 \zeta_1, \\
\hat{\alpha}_3 &= \beta_0^2 \hat{\alpha}_1^3 \zeta_1^2 + \hat{\alpha}_1^2 (\beta_{10} \hat{\alpha}_1 + \beta_{01} \hat{y}_1) \zeta_1 + \beta_0 \hat{\alpha}_1 (c_5 \hat{\alpha}_1^2 + c_6 \hat{\alpha}_1 \hat{y}_1 + c_7 \hat{y}_1^2), \\
\hat{y}_1 &= \frac{\gamma_{10} - \beta_0}{\beta_0 \gamma_{01} \hat{L} + K (\gamma_{10} - \beta_0) \hat{L}^\nu}, \\
\hat{y}_2 &= \hat{y}_1 (\gamma_{10} \hat{\alpha}_1 + \gamma_{01} \hat{y}_1) \zeta_1 + \hat{y}_1 (c_3 \hat{\alpha}_1 + c_4 \hat{y}_1), \\
\hat{y}_3 &= \hat{y}_1 (\gamma_{01} \hat{y}_1^2 + \frac{3}{2} \gamma_{10} \hat{y}_1 \hat{\alpha}_1 + \frac{\gamma_{10} (\gamma_{10} + \beta_0)}{2 \gamma_{01}} \hat{\alpha}_1^2) \zeta_1^2 \\
&+ \hat{y}_1 (c_8 \hat{y}_1^2 + c_9 \hat{y}_1 \hat{\alpha}_1 + c_{10} \hat{\alpha}_1^2) \zeta_1 + \hat{y}_1 (c_{11} \hat{y}_1^2 + c_{12} \hat{y}_1 \hat{\alpha}_1 + c_{13} \hat{\alpha}_1^2),
\end{aligned}
$$

$\ldots,$

$$\zeta_1 := c_1 \ln \hat{\alpha}_1 + c_2 \ln \hat{y}_1.$$

Series of leading, subleading, etc series of logarithms of couplings (powers of ζ_1) reproduce solutions α_ℓ and y_ℓ!

"Vertical" summation

Series of logarithms of couplings reproduce the above obtained solutions:

$$\hat{\alpha}_1^{(1)} = \frac{\hat{\alpha}_1}{1-\beta_0\hat{\alpha}_1\zeta_1},$$

$$\hat{y}_1^{(1)} = \frac{(\gamma_{10}-\beta_0)\hat{\alpha}_1\hat{y}_1}{-\gamma_{01}\hat{y}_1(1-\beta_0\hat{\alpha}_1\zeta_1)+(\hat{\alpha}_1(\gamma_{10}-\beta_0)+\gamma_{01}\hat{y}_1)(1-\beta_0\hat{\alpha}_1\zeta_1)^{\nu}},$$

$$\hat{\alpha}_\ell^{(1)} = \hat{\alpha}_\ell\left[\hat{\alpha}_1\to\hat{\alpha}_1^{(1)},\ \hat{y}_1\to\hat{y}_1^{(1)},\ \ln\hat{\alpha}_1\to\ln\hat{\alpha}_1^{(1)}/\hat{\alpha}_1,\ \ln\hat{y}_1\to\ln\hat{y}_1^{(1)}/\hat{y}_1\right],$$

$$\hat{y}_\ell^{(1)} = \hat{y}_\ell\left[\hat{\alpha}_1\to\hat{\alpha}_1^{(1)},\ \hat{y}_1\to\hat{y}_1^{(1)},\ \ln\hat{\alpha}_1\to\ln\hat{\alpha}_1^{(1)}/\hat{\alpha}_1,\ \ln\hat{y}_1\to\ln\hat{y}_1^{(1)}/\hat{y}_1\right],$$

Just the change of the arguments!
But that is only the first step...

"Vertical" summation

After the resummation we got the same functions $\Rightarrow$ we can resummate more, and more!

n-th step of resummation:

$$\hat{\alpha}_1^{(n)} = \frac{\hat{\alpha}_1^{(n-1)}}{1-\beta_0\hat{\alpha}_1^{(n-1)}\zeta_n},$$

$$\hat{y}_1^{(n)} = \frac{(\gamma_{10}-\beta_0)\hat{\alpha}_1^{(n-1)}\hat{y}_1^{(n-1)}}{-\gamma_{01}\hat{y}_1^{(n-1)}(1-\beta_0\hat{\alpha}_1^{(n-1)}\zeta_n)+(\hat{\alpha}_1^{(n-1)}(\gamma_{10}-\beta_0)+\gamma_{01}\hat{y}_1^{(n-1)})(1-\beta_0\hat{\alpha}_1^{(n-1)}\zeta_n)^{\nu}}$$

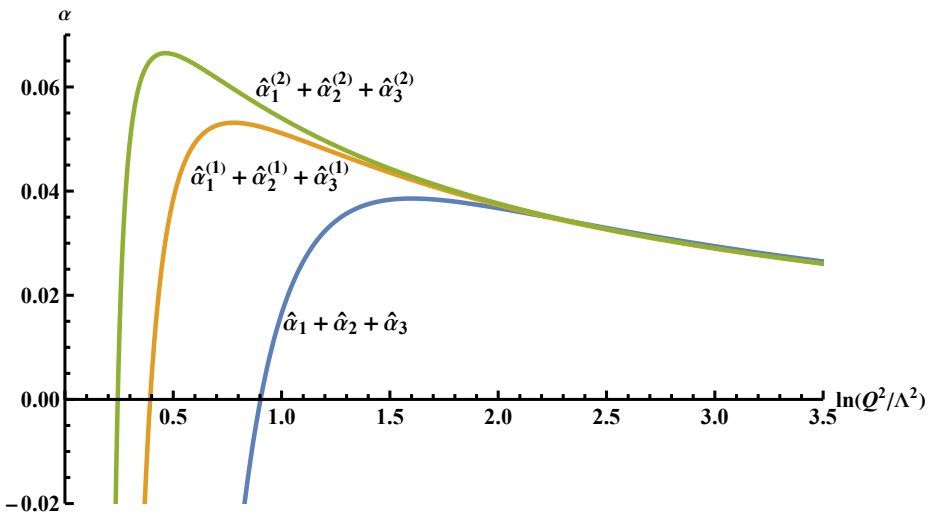
$$\hat{\alpha}_\ell^{(n)} = \Xi_n^\ell\hat{\alpha}_\ell\left[\hat{\alpha}_1^{(n)}/\Xi_n,\,\hat{y}_1^{(n)}/\Xi_n\right],$$

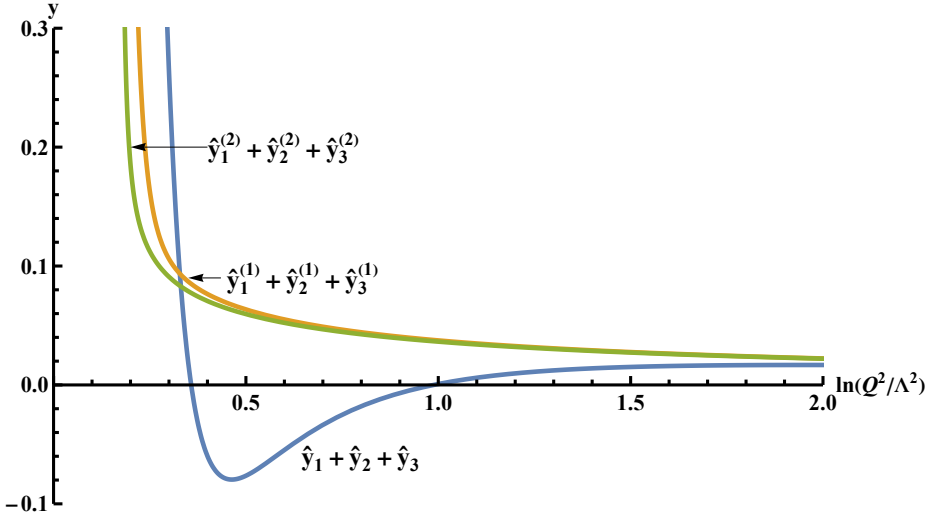
$$\hat{y}_\ell^{(n)} = \Xi_n^\ell\hat{y}_\ell\left[\hat{\alpha}_1^{(n)}/\Xi_n,\,\hat{y}_1^{(n)}/\Xi_n\right],$$

$$\zeta_n = c_1\ln(\hat{\alpha}_1^{(n-1)}/\hat{\alpha}_1^{(n-2)})+c_2\ln(\hat{y}_1^{(n-1)}/\hat{y}_1^{(n-2)}),\quad \Xi_n = \left((\hat{\alpha}_1^{(n-1)})^{c_1}(\hat{y}_1^{(n-1)})^{c_2}\right)^{\frac{1}{c_1+c_2}}$$

"Vertical" summation

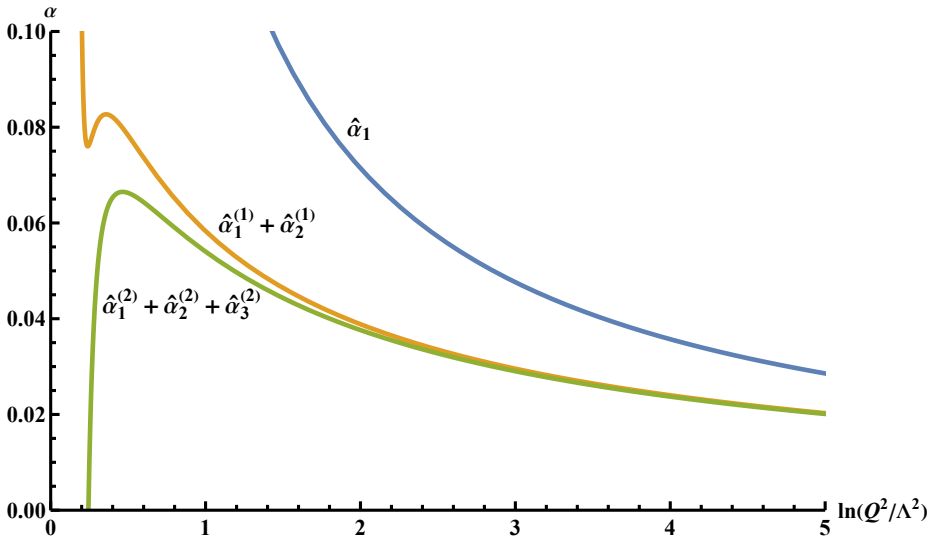
Resummation allows to capture more terms of the expansion and to improve the solutions near the IR pole, extending the region where we trust our solutions!

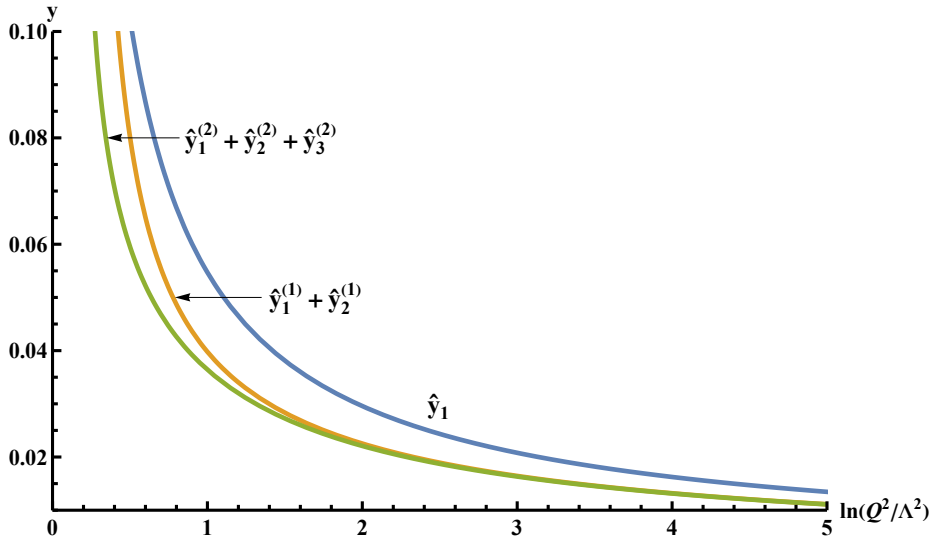




Best solution for N known loops

$$\boxed{\bar{\alpha} = \sum_{\ell=1}^{N} \hat{\alpha}_\ell^{(N-1)}, \quad \bar{y} = \sum_{\ell=1}^{N} \hat{y}_\ell^{(N-1)}}$$





Conclusions

- You do not need to solve approximate equations precisely
- Solving linearized equations allows extracting needed series of a solution ($\sum_{n=\ell}^{\infty}(\alpha_0 + y_0)^n L^{n-\ell}$)
- Solving linearized equations results in simple analytical forms of the solutions
- By solving linearized equations one can switch to RG-invariant formulation, as well as calculate further loop contributions for the RG-invariants
- Solutions of the linearized equations allow the "vertical" resummation which is performed by simple change of argument and which improves the solutions by capturing more terms of the expansion

Thank you for your attention!