Radiative corrections of order $\alpha(Z\alpha)^6$ in the energy spectrum of muonic hydrogen

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Introduction

Precision studies of the energy spectra of muonic atoms currently encompass the following key objectives:

- The Lamb shift in muonic hydrogen
- The Lamb shift in muonic helium ions
- Hyperfine splitting of the ground state in muonic hydrogen
- Fine and hyperfine structure of the spectra in muonic lithium, beryllium, and boron ions
- Fine-structure intervals (such as the 1S–2S transition) in muonic hydrogen and muonic helium ions

Precision theoretical investigation of energy spectra entails calculating high-order corrections in α for energy intervals, consistent with the experimental precision achieved in measuring energy levels. This enables the determination of fundamental physical constants—such as nuclear charge radii and the Rydberg constant—with high accuracy.

The aim of the work: to study radiation corrections in the energy spectrum of the simplest muonic atoms.
The following **research objectives** were identified:

1. **Construct** the particle interaction operator based on three-photon exchange amplitudes;
2. **Construct** the particle interaction operator based on three-photon exchange amplitudes taking into account corrections for *self-energy, vacuum polarization, and vertex corrections*;
3. **Perform** a calculation of the contribution of three-photon exchange amplitudes with radiative corrections to the Lamb shift of muonic hydrogen, taking into account the structure of the nucleus.

A. Antognini, F. Hagelstein, V. Pascalutsa, The Proton Structure in and out of Muonic Hydrogen, Ann. Rev. Nucl. and Part. Science **72**, 389 (1922).

The Proton Radius Puzzle

"The proton radius puzzle" refers to the discrepancy between the values of the proton charge radius, r_p, obtained from experiments with electronic hydrogen and those from experiments with muonic hydrogen.

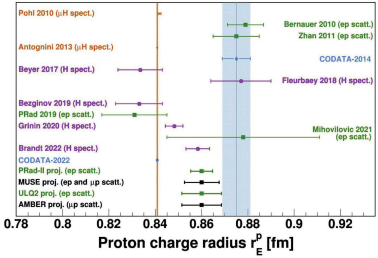
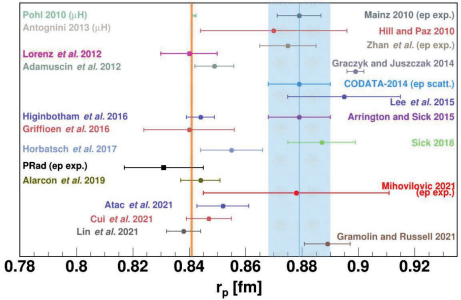


Figure: The latest r_p results from atomic spectroscopy measurements together with muonic spectroscopy results, as well as the CODATA-2014 recommended value based on ordinary atomic spectroscopy and the most recent result from electron scattering

The Proton Radius Puzzle

The r_p values determined from electron scattering experiments (performed after 2010) together with the results from various (re-)analyses of $e - p$ scattering data.



H. Gao, The charge radius of the proron, Journal of Subatomic Particles and Cosmology, 6, 100495 (2026).

The Proton Radius Puzzle

- Early measurements of the proton charge radius (spectroscopy and $e - H$ scattering) $\rightarrow$ $r_p = 0.8775(51)$ fm

  
  P.J. Mohr, D.B. Newell, B.N. Taylor, CODATA recommended values of the fundamental physical constants: 2014, Rev. Mod. Phys. **88**, 035009 (2016).

- Measurement of the Lamb shift in $\mu H \rightarrow r_p = 0.84184(67)$ fm

  
  R. Pohl, A. Antognini, F. Nez et al., Nature **466**, 213 (2010)

- Measurement of the Lamb shift in $\mu H \rightarrow r_p = 0.84087(39)$ fm

  
  A. Antognini et al., Science **339**, 417 (2013)

- Measuring the frequency of the $2S - 4P$ transition in $eH \rightarrow r_p = 0.8335(95)$ fm

  
  A. Beyer, et al., Science **358**, 79–85 (2017)

The Proton Radius Puzzle

- Measurement of the $1S - 3S$ transition frequency in $eH \rightarrow r_p = 0.877(13)$ fm

  H. Fleurbaey et al. Phys. Rev. Lett. **120**, 183001 (2018)

- MUSE collaboration, The MUon Scattering Experiment (MUSE) at the Paul Scherrer Institute,

  MUSE collaboration, PoS NuFACT2018 (2018) 136.

  The MUon Scattering Experiment (MUSE) at PSI is a scattering experiment that is ongoing aiming to resolve the r_p puzzle. It measures the charge radius of the proton from the $\mu^{\pm} - p$ and $e^{\pm} - p$ elastic scattering cross sections.

- The AMBER Experiment at CERN: r_p measurement in high-energy low-Q2 elastic μp scattering at the M2 beam line of the CERN Super Proton Synchrotron (SPS)

  B. Adams et al., COMPASS++/AMBER: Proposal for Measurements at the M2 beam line of the CERN SPS Phase-1: 2022-2024.

Relevance of the study

In recent years, there has been a significant increase in the accuracy of measuring the fine and hyperfine structure of the spectrum of the simplest atoms. Thus, the J-PARC MuSEUM collaboration measured the hyperfine splitting in muonium with an accuracy of up to 4 KHz and plans to further increase the accuracy to 1 Hz (1 ppb). The CREMA collaboration measured the Lamb shift in muonic hydrogen and obtained the value $\Delta E_{LS}^{exp} = 206.2949(32)$ meV, the theoretical value was equal to $\Delta E_{LS}^{th} = 209.9779(49) - 5.2262 r_p^2 + 0.0347 r_p^3$ meV.

1. Kanda S. et al., New precise spectroscopy of the hyperfine structure in muonium with a high-intensity pulsed muon beam, Phys. Lett. B 815, 136154 (2021),
2. Antognini A. et al., Theory of the 2S-2P Lamb shift and 2S hyperfine splitting in muonic hydrogen, Annals Phys. 331 127-145 (2013).

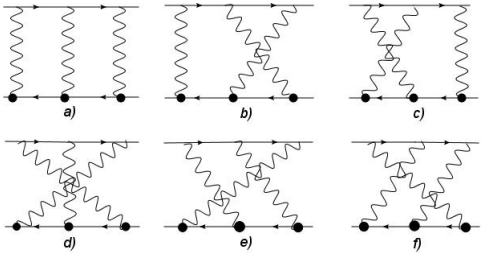
This increase in experimental accuracy necessitates theoretical calculations of contributions of multi-loop Feynman amplitudes. We continue our investigation of radiative corrections involving nuclear structure, building on our previous work:

R.N. Faustov, A.P. Martynenko, F.A. Martynenko, V.V. Sorokin, Phys. Lett. B 775, 79 (2017)

Corrections of order $(Z\alpha)^6$ from three-photon exchange amplitudes

G.T. Bodwin, D.R. Yennie, M.A. Gregorio, Recoil Effects in the Hyperfine Structure of QED Bound States, Rev. Mod. Phys. **57**, 723 (1985)

K. Pachucki, V. Patkos, V. Yerokhin, Three-photon-exchange nuclear structure correction in hydrogenic systems, Phys. Rev. A 97, 062511 (2018).



Let us consider the amplitudes of three-photon interaction taking into account the structure of the nucleus. These diagrams will contribute to the energy levels at the order of $(Z\alpha)^6$. The bold dot denotes the proton vertex operator.

The interaction amplitude corresponding to the first diagram:

$$M_1 = -\frac{(Z\alpha)^3}{4\pi^5}\int d^4p d^4p' \frac{1}{\hat{p}^2}\frac{1}{\hat{p}'^2}\frac{F_1(\hat{p}^2)}{(\hat{p}-\hat{p}')^2}\frac{F_1(\hat{p}'^2)}{D_\mu(-\hat{p})D_\mu(-\hat{p}')}\frac{F_1((\hat{p}-\hat{p}')^2)}{D_p(\hat{p})D_p(\hat{p}')}\times \tag{1}$$

$$\times Tr[\hat{\epsilon}\frac{1+\gamma_0}{2\sqrt{2}}\gamma_\mu(\hat{q}_1-\hat{p}'+m_1)\gamma_\nu(\hat{p}_1-\hat{p}+m_1)\gamma^\lambda\frac{1+\gamma_0}{2\sqrt{2}}\hat{\epsilon}\gamma_\lambda(-\hat{p}_2-\hat{p}+m_2)\gamma^\nu(-\hat{q}_2-\hat{p}'+m_2)\gamma^\mu],$$

Here, F_1 is the charge form factor of the nucleus, p, p' are the loop momenta, D_μ is the muon propagator, and D_p is the proton propagator.

$$[u_i(0)\overline{v}_k(0)]_{s=0} = \frac{1+\gamma^0}{2\sqrt{2}}\gamma^5, \qquad [u_i(0)\overline{v}_k(0)]_{s=1} = \frac{1+\gamma^0}{2\sqrt{2}}(\epsilon^\mu\gamma^\mu), \tag{2}$$

The traces of γ-matrices were calculated using the FORM package.

J. Kuipers, T. Ueda, J. Vermaseren A. M., and J. Vollinga, FORM version 4.0, Comput. Phys. Commun. 184, 2013, 1453.

To analyze the contribution to the Lamb shift, we take into account the leading-order contribution in terms of m_2. It turns out that this part is the same in all integrals.

$$N = m_2^2[\frac{3}{2}p_0p_0' - \frac{3}{4}pp' - \frac{3}{2}m_1p_0 - \frac{3}{2}m_1p_0' + 3m_1^2]. \tag{3}$$

The integrals themselves will differ only in the proton propagators. Then, when they are summed, the following expression will appear:

$$\frac{1}{D_p(p)D_p(p')} + \frac{1}{D_p(p')D_p(p'-p)} + \frac{1}{D_p(p)D_p(p-p')} + \tag{4}$$

$$+\frac{1}{D_p(-p)D_p(-p')} + \frac{1}{D_p(-p')D_p(p-p')} + \frac{1}{D_p(-p)D_p(p'-p)} = (\frac{-2\pi i}{2m_2})^2\delta(p_0')\delta(p_0).$$

To identify the leading order, we arrange the form factors in a series: $(r_p^2 = 6\frac{F_1'(0)}{F_1(0)})$

$$F(p^2)F(p'^2)F((p-p')^2) = 1 - \frac{r_p^2}{6}[\vec{p}^{\,2} + \vec{p}^{\,\prime 2} + (\vec{p} - \vec{p}^{\,\prime})^2], \text{where } r_p^2 \text{ is the charge radius of the proton.} \tag{5}$$

After the specified transformations, we obtain the following integral:

$$I = \frac{(Z\alpha)^3}{4\pi^3}\frac{16}{3}\frac{(-r_p^2)}{6}\int d\vec{p}\,d\vec{p}\,'\frac{[\vec{p}^{\,2}+\vec{p}^{\,\prime 2}+(\vec{p}-\vec{p}^{\,\prime})^2][-3/4\,\vec{p}\,\vec{p}^{\,\prime}+3m_1^2]}{(\vec{p}^{\,2})(\vec{p}^{\,\prime 2})(\vec{p}-\vec{p}^{\,\prime})^2(\vec{p}^{\,2}+w^2)(\vec{p}^{\,\prime 2}+w^2)}. \tag{6}$$

The expansion of the potential in perturbation theory takes the form:

$$V^{(1)} = T^{(1)}, \tag{7}$$

$$V^{(2)} = T^{(2)} - T^{(1)}G_f T^{(1)}, \tag{8}$$

$$V^{(3)} = T^{(3)} - T^{(2)}G_f T^{(1)} - T^{(1)}G_f T^{(2)} + T^{(1)}G_f T^{(1)}G_f T^{(1)} \tag{9}$$

The total contribution to the energy spectrum is determined by the sum of the amplitude and iteration terms of the quasi-potential. Let us consider the iteration term of the potential:

$$\Delta V_{iter} = T^{(1)}G_f T^{(1)}G_f T^{(1)}, \tag{10}$$

where $T^{(1)} = -\frac{Ze^2F_1}{\vec{p}^{\,2}}$ is the amplitude of the single-photon interaction.,
$G_f(\vec{p},\vec{q}) = (2\pi)^3\delta(\vec{p}-\vec{q})\frac{-2\mu}{(w^2+\vec{p}^{\,2})}$ is the free two-particle propagator.

Expanding the form factors at small momenta, we write the iterative term of the potential:

$$\Delta V_{iter} = -\frac{(Z\alpha)^3}{\pi^3} 4m_1^2 \frac{(-r_p^2)}{6} \int d\overrightarrow{p}\, d\overrightarrow{p}' \frac{[\overrightarrow{p}^2 + \overrightarrow{p}'^2 + (\overrightarrow{p} - \overrightarrow{p}')^2]}{(\overrightarrow{p}^2)(\overrightarrow{p}'^2)(\overrightarrow{p} - \overrightarrow{p}')^2(\overrightarrow{p}^2 + w^2)(\overrightarrow{p}'^2 + w^2)}. \quad (11)$$

The main integral, taking into account the iterative term, will contribute to the Lamb shift. After calculating the resulting expression, we will get the following expression:
One of the integrals over the loop momentum is finite, while the second integral is logarithmically divergent. The reason for this divergence lies in the expansion of the form factors that was employed. To evaluate it, a momentum cutoff Λ is used.

$$\Delta E^{LS} = \frac{2(Z\alpha)^6 r_p^2 \mu^3}{3n^3}[-ln\frac{w}{\Lambda} - \frac{5}{8} - ln2] = 0.0011\ meV. \quad (12)$$

The integration region for these integrals will be cut off by momentum for the calculation using the parameter $\Lambda = \sqrt{12/r_p^2}$ at the hadronic scale.

The proton vertex operator is defined by the value:

$$\Gamma^{\mu} = \gamma^{\mu} F_1 - \frac{1}{2m_2} \sigma^{\mu\nu} k_{\nu} F_2, \tag{13}$$

where F_1 is the electric form factor, F_2 is the magnetic form factor.
Let's introduce the following notation:

$$\begin{gathered} F_1(-p^2) = f_1; \quad F_2(-p^2) = g_1; \\ F_1(-(p-p')^2) = f_2; \qquad F_2(-(p-p')^2) = g_2; \\ F_1(-p'^2) = f_3; \qquad F_2(-p'^2) = g_3. \end{gathered} \tag{14}$$

The interaction amplitudes for calculating the HFS will be constructed using the same diagrams as in Fig. 2, but when calculating the contribution to the energy levels, it is necessary to take into account the magnetic form factors F_2. Thus, the numerator of the interaction amplitudes will have the following structure:

$$N = g_1 f_2 f_3(...) + f_1 g_2 f_3(...) + f_1 f_2 g_3(...) + f_1 f_2 f_3(...). \tag{15}$$

After analyzing the numerators, we noticed that the numerators differ only in the terms containing the factor $f_1 f_2 f_3$. Then our reasoning is divided into two parts. In the first part, we will focus on the identical terms. In this case, when summing the amplitudes, we will obtain δ-functions, as was shown in (3). The numerator, taking into account the δ-functions, will look like this:

$$g_1 f_2 f_3(-2m_1 \overrightarrow{p}^2) + f_1 g_2 f_3(-2m_1(\overrightarrow{p}^2 + \overrightarrow{p}'^2 - 2\overrightarrow{p}\,\overrightarrow{p}')) + f_1 f_2 g_3(-2m_1 \overrightarrow{p}'^2). \tag{16}$$

In the second part of the reasoning, we need to analyze the terms with the factor $f_1 f_2 f_3$. We will also identify the identical and differing parts among them. Let us write down the identical parts of the numerators of amplitudes 1–6, which have the factor $f_1 f_2 f_3$ and do not contain the factors p_0:

$$-4m_1 f_1 f_2 f_3(\overrightarrow{p}^2 + \overrightarrow{p}'^2 - \overrightarrow{p}\,\overrightarrow{p}'), \tag{17}$$

Let us write down the differing parts of the numerators of the amplitudes $M_1 - M_6$, which have the factor $f_1 f_2 f_3$:

$$M_2; M_5 : -6m_1 {p_0'}^2 - 6m_1 p_0 p_0' + 5p_0' pp' - 2p_0 {p_0'}^2, \tag{18}$$

$$M_3; M_6 : -6m_1 p_0' p_0 - 6m_1 p_0^2 + 5p_0 pp' - 2p_0^2 p_0'. \tag{19}$$

Note that in expression (17), each term contains p_0', while in (18) it contains p_0. We will need this feature for further analysis. Let us write the sum $M_2 + M_5$ taking into account the factor p_0'. This factor removes the singularity, and the $i0$ disappears in the corresponding propagator.

$$p_0'\left[\frac{1}{(p_0' - p_0 + i0)(p_0' + i0)} + \frac{1}{(p_0 - p_0' + i0)(-p_0' + i0)}\right] = \tag{20}$$

$$= \frac{1}{(p_0' - p_0 + i0)} - \frac{1}{(p_0 - p_0' + i0)} = \frac{2}{(p_0' - p_0 + i0)} + 2\pi i\delta(p_0' - p_0).$$

Using a similar transformation for $M_3 + M_6$ and applying this method to calculate the sum of amplitudes $M_2 + M_5 + M_3 + M_6$ with numerators (17) and (18), we will obtain a result equal to zero.

Let's write down the integral, discarding all elements of the numerator that contribute nothing.

$$M^{CTC} = -\frac{(Z\alpha)^3}{4\pi^5}\frac{16}{3}\int d^4p d^4p' \frac{1}{(p^2-w^2)^2(p'^2-w^2)^2(p-p')^2}\frac{(-2\pi i)^2\delta(p_0)\delta(p'_0)}{4m_2^2} \tag{21}$$

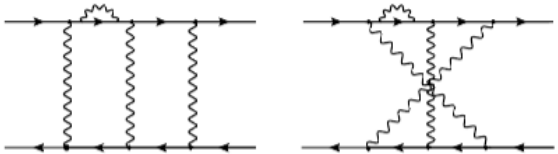
$$\times(-2m_1)m_2[g_1f_2f_3(\overrightarrow{p}^2) + f_1g_2f_3(\overrightarrow{p}^2 + \overrightarrow{p}'^2 - 2\overrightarrow{p}\,\overrightarrow{p}') + f_1f_2g_3(\overrightarrow{p}'^2) + f_1f_2f_3(2\overrightarrow{p}^2 + 2\overrightarrow{p}'^2 - 2\overrightarrow{p}\,\overrightarrow{p}')].$$

Arranging the form factors in a row, we get:

$$M^{hfs}_{1-6} = -\frac{8(Z\alpha)^3}{3\pi^3}\frac{m_1}{m_2}\int d\vec{p}d\vec{p}'\frac{\mu_p(-\frac{r_p^2}{6})[\overrightarrow{p}^2 + \overrightarrow{p}'^2 + (\overrightarrow{p}-\overrightarrow{p}')^2][\overrightarrow{p}^2 + \overrightarrow{p}'^2 + (\overrightarrow{p}-\overrightarrow{p}')^2]}{(\vec{p}^2+w^2)^2(\vec{p}'^2+w^2)^2(\vec{p}-\vec{p}')^2}. \tag{22}$$

Taking into account the iterative term, which is determined by formula (9), completely eliminates the main integral (21). Thus, these diagrams do not contribute to the HFS of muonic hydrogen at the leading order.

Calculation of corrections of order $\alpha(Z\alpha)^6$ to the muon self-energy, taking into account the structure of the nucleus.



The data in the diagrams differ from the previous ones due to the radiative photon in the muon. The correction for it is called the self-energy correction. It is calculated using the Fried–Yennie gauge and, taking renormalization into account, is written as follows:

$$\Sigma^R(p_1-p)=(\hat{p}_1-\hat{p}-m_1)^2\left(\frac{-3\alpha}{4\pi}\right)(\hat{p}_1-\hat{p})\int_0^1\frac{(1-x)dx}{m_1^2-x(m_1^2+p^2)+2xm_1p_0} \qquad (23)$$



Let's write down the sum of the amplitudes from all 6 diagrams:

$$M^{LS}_{1-6,se} = -\frac{9\alpha(Z\alpha)^3}{32\pi^4}\int d^4p \int d^4p'\delta(p_0)\delta(p'_0)\frac{F_1(\mathbf{p}^2)}{\mathbf{p}^2}\frac{F_1(\mathbf{p'}^2)}{\mathbf{p'}^2}\frac{F_1((\mathbf{p}-\mathbf{p'})^2)}{(\mathbf{p}-\mathbf{p'})^2} \tag{24}$$
$$\int_0^1 \frac{(1-x)dx}{(x\mathbf{p}^2)+(1-x)b_1^2}\frac{(\mathbf{pp'})}{(\mathbf{p'}^2+1)}$$

Here, the momenta are nondimensionalized using W, and a redefinition $b_1 = m_1/W$ is introduced. Expanding the form factors into a series and expanding the sum in the numerator into a sum of integrals, we obtain the following non-zero integrals:

$$I_1 = \int d\mathbf{p}d\mathbf{p'}\frac{(\mathbf{pp'})}{\mathbf{p'}^2(\mathbf{p'}^2+1)(\mathbf{p}-\mathbf{p'})^2}\frac{\mathbf{p}^2(ln(\mathbf{p}^2/b_1^2)-1)+b_1^2}{(\mathbf{p}^2-b_1^2)^2} \tag{25}$$

$$I_2 = \int d\mathbf{p}d\mathbf{p'}\frac{(\mathbf{pp'})}{\mathbf{p}^2(\mathbf{p'}^2+1)(\mathbf{p}-\mathbf{p'})^2}\frac{\mathbf{p}^2(ln(\mathbf{p}^2/b_1^2)-1)+b_1^2}{(\mathbf{p}^2-b_1^2)^2} \tag{26}$$

The integration region for these integrals will be cut off by momentum for the calculation using the parameter $\Lambda = \sqrt{12/r_N^2}$ at the hadronic scale.

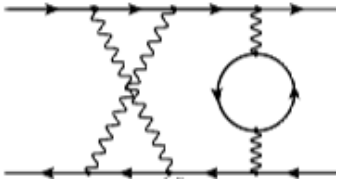
When calculated, these integrals yield the following result:

$$I_1 = 2\pi^4(ln^2(b_2/b_1) + ln(b_2/b_1) + \pi^2/6) \tag{27}$$

$$I_2 = 2\pi^4(ln^2(b_2/b_1) + 2ln(b_2/b_1) + \pi^2/3) \tag{28}$$

where $b_2 = \Lambda/W$.
As a result, the contribution of the muon's own energy to the shift of S-levels is determined by the formula:

$$\Delta E^{LS}_{str,se} = \frac{3\alpha(Z\alpha)^6\mu^3 r_N^2}{32\pi n^3}(2ln^2(\Lambda/m_1) + 3ln(\Lambda/m_1) + \pi^2/2) \tag{29}$$



The contribution of vacuum polarization is determined by the replacement of the photon propagator:

$$\frac{1}{p^2} \to -\frac{\alpha}{\pi}\int_0^1 \frac{x^2(1-x^2/3)dx}{\mathbf{p}^2(1-x^2)+4m_e^2} = \tag{30}$$

$$= -\frac{\alpha}{\pi p^5}[4m_e^2 p - 5p^3/3 - (4m_e^2 - 2p^2)\sqrt{4m_e^2+p^2}\,arcsinh(\frac{p}{2m_e})]$$

The contribution to the Lamb shift, calculated from diagrams with polarization of the end photon, is determined by the expression:

$$\Delta E_{vp}^{(1)} = \frac{\alpha(Z\alpha)^6\mu^3 r_p^3}{48\pi^5 n^3}\int\frac{d\mathbf{p}}{\mathbf{p}^2+1}\int\frac{d\mathbf{p'}}{\mathbf{p'}^2(\mathbf{p'}^2+1)}\frac{(\mathbf{p}\ \mathbf{p'})(\mathbf{p}^2+\mathbf{p'}^2)}{(\mathbf{p}-\mathbf{p'})^2} \tag{31}$$

$$\frac{1}{p^5}[b_1^2 p - 5p^3/3 - (b_1^2-2p^2)\sqrt{b_1^2+p^2}\,arcsinh(\frac{p}{b_1})]$$

where the notation $b_1 = 2m_e/W$ is introduced. To calculate the integral, we divide it into two parts.

$$J_1 = \int\frac{d\mathbf{p}}{\mathbf{p}^2+1}\int\frac{d\mathbf{p'}}{\mathbf{p'}^2(\mathbf{p'}^2+1)}\frac{(\mathbf{p}\ \mathbf{p'})}{(\mathbf{p}-\mathbf{p'})^2}\frac{1}{p^3}[b_1^2 p - 5p^3/3 - (b_1^2-2p^2)\sqrt{b_1^2+p^2}\,arcsinh(\frac{p}{b_1})] \tag{32}$$

$$J_2 = \int\frac{d\mathbf{p}}{\mathbf{p}^2+1}\int\frac{d\mathbf{p'}}{(\mathbf{p'}^2+1)}\frac{(\mathbf{p}\ \mathbf{p'})}{(\mathbf{p}-\mathbf{p'})^2}\frac{1}{p^5}[b_1^2 p - 5p^3/3 - (b_1^2-2p^2)\sqrt{b_1^2+p^2}\,arcsinh(\frac{p}{b_1})] \tag{33}$$

We also use momentum cut off with the parameter Λ.

After integration, we obtain:

$$J_1 = 2\pi^4[ln^2(\frac{\Lambda}{W}) + 4ln(\frac{\Lambda}{W})(ln(\frac{2m_e}{W}) - ln(2)/2 + 5/12)] \tag{34}$$

$$J_2 = \frac{2\pi^4}{9} ln(\frac{\Lambda}{W})(3ln(\frac{\Lambda}{W}) + 6ln(\frac{2m_e}{W}) - 6ln(2) + 8) \tag{35}$$

Ultimately, the contribution to the Lamb shift is determined by the expression:

$$\Delta E_{vp}^{(1)} = \frac{\alpha(Z\alpha)^6\mu^3 r_p^3}{12\pi n^3} ln(\frac{\Lambda}{W})(2ln(\frac{\Lambda}{W}) + 7ln(\frac{2m_e}{W}) - 4ln(2) + 23/6) \tag{36}$$

To calculate the diagram with the polarization of the intermediate photon, we use the following form of the photon propagator with vacuum polarization:

$$\frac{1}{(\mathbf{p}-\mathbf{p'})^2} \to -\frac{\alpha}{3\pi}\int_0^\infty \frac{\rho(\xi)d\xi}{[(\mathbf{p}-\mathbf{p'})^2+4m_e^2\xi^2]}, \quad \rho(\xi)=\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4} \tag{37}$$

The contribution to the Lamb shift will be determined by the following expression:

$$\Delta E_{vp}^{(2)} = \frac{\alpha(Z\alpha)^6\mu^3 r_p^3}{95\pi^5 n^3}\int\frac{d\mathbf{p}}{\mathbf{p}^2(\mathbf{p}^2+1)}\int\frac{d\mathbf{p'}}{\mathbf{p'}^2(\mathbf{p'}^2+1)}\int_0^\infty \rho(\xi)d\xi\frac{(\mathbf{p}\ \mathbf{p'})[4\mathbf{p'}^2-2(\mathbf{p}\ \mathbf{p'})]}{[(\mathbf{p}-\mathbf{p'})^2+4m_e^2\xi^2]} \tag{38}$$

Also, by breaking the integral down into a sum of integrals, we obtain:

$$K_1 = \int \frac{d\mathbf{p}}{\mathbf{p}^2(\mathbf{p}^2+1)} \int \frac{d\mathbf{p'}}{(\mathbf{p'}^2+1)} \int_0^\infty \rho(\xi) d\xi \frac{(\mathbf{p}\ \mathbf{p'})}{[(\mathbf{p}-\mathbf{p'})^2 + 4m_e^2\xi^2]} \tag{39}$$

$$K_2 = \int \frac{d\mathbf{p}}{\mathbf{p}^2(\mathbf{p}^2+1)} \int \frac{d\mathbf{p'}}{\mathbf{p'}^2(\mathbf{p'}^2+1)} \int_0^\infty \rho(\xi) d\xi \frac{(\mathbf{p}\ \mathbf{p'})^2}{[(\mathbf{p}-\mathbf{p'})^2 + 4m_e^2\xi^2]} \tag{40}$$

Direct calculation of these integrals is difficult, so we apply the Fourier transform to the propagators in these integrals.

$$\frac{4\pi\mathbf{p}}{\mathbf{p}^2} = \int \exp(-i\mathbf{pr}) \frac{i\mathbf{r}}{r^3} d\mathbf{r}; \quad \frac{4\pi\mathbf{p'}}{\mathbf{p'}^2+1} = \int \exp(-i\mathbf{p'r'}) \frac{i\mathbf{r'}}{r'^3} [1+r'] \exp(r') d\mathbf{r'}; \tag{41}$$

$$\frac{4\pi}{\mathbf{k}^2+m^2} = \int \exp(-i\mathbf{ks}) \frac{1}{s} \exp(-ms) d\mathbf{s}$$

After applying these transformations, the integrals become significantly simpler.

For integration by coordinates, it is also necessary to trim the integration region from below using the parameter $(\frac{\Lambda}{W})^{-1}$.

$$K_1 = -K_2 = -2\pi \ln \frac{W}{\Lambda}[\ln \frac{W}{\Lambda} + 2(\ln \frac{2m_e}{W} - \ln 2 + \gamma + 5/6)] \tag{42}$$

Ultimately, the contribution of vacuum polarization in the average exchanged photon to the Lamb shift is given by the expression:

$$\Delta E_{vp}^{(2)} = -\frac{\alpha(Z\alpha)^6 \mu^3 r_p^6}{4\pi n^3} \ln \frac{W}{\Lambda}(\frac{1}{2} \ln \frac{W}{\Lambda} + \ln \frac{2m_e}{W} - \ln 2 + 5/6 + \gamma) \tag{43}$$

Numerical results of calculation

$$\Delta E^{LS}_{str} = \frac{2(Z\alpha)^6 r_p^2 \mu^3}{3n^3}[-ln\frac{w}{\Lambda} - \frac{5}{8} - ln2] = 0.0011\ meV, \tag{44}$$

$$\Delta E^{LS}_{str,se} = \frac{3\alpha(Z\alpha)^6 \mu^3 r_p^2}{32\pi n^3}(2ln^2(\frac{\Lambda}{m_1}) + 3ln(\frac{\Lambda}{m_1}) + \frac{\pi^2}{2}) = 0.000457\ \mu eV, \tag{45}$$

$$\Delta E^{(LS,1)}_{str,vp} = \frac{\alpha(Z\alpha)^6 \mu^3 r_p^2}{12\pi n^3} ln(\frac{\Lambda}{W})(2ln(\frac{\Lambda}{W}) + 7ln(\frac{2m_e}{W}) - 4ln(2) + \frac{23}{6}) = 0.007208\ \mu eV, \tag{46}$$

$$\Delta E^{(LS,2)}_{str,vp} = \frac{\alpha(Z\alpha)^6 \mu^3 r_p^2}{4\pi n^3} \ln\frac{\Lambda}{W}(\frac{1}{2}\ln\frac{W}{\Lambda} + \ln\frac{2m_e}{W} - \ln 2 + \frac{5}{6} + \gamma) = -0.002930\ \mu eV. \tag{47}$$

$$\Delta E^{(LS,1)}_{str,vp} + \Delta E^{(LS,2)}_{str,vp} + \Delta E^{LS}_{str,se} = 0.004735\ \mu eV, \tag{48}$$

$$\Delta E^{LS}_{exp} = 206.2949(32)\ meV. \tag{49}$$